T-martingales, size-biasing and tree polymer cascades

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1 Introduction

Directed lattice polymers on the $d+1$ dimensional integer lattice are modeled by (random) distributions of graphs of polygonal paths in $\mathbf{N} \times \mathbf{Z}^d$ for which the horizontal coordinate serves to direct the path as a self-avoiding chain of connected monomers.

Tree polymer models were considered early on by Bolthausen (1991) as a special framework in which to illustrate certain L^2-martingale methods introduced to analyze directed lattice polymers. In this paper we will use the term **lattice polymer** in reference to the directed polygonal paths on the $d+1$ dimensional integer lattice, and **tree polymer** for the case of polygonal paths of a binary tree.

While the primary focus of polymer research is aimed at low dimensional lattice polymer models, where sharp results are rare, the tree polymer is important for testing lattice methods because sharp results are often possible to obtain for tree paths. In fact one can demand sharp results and precise cutoffs of tree polymer theory, whereas this seems a less realistic requirement of lattice polymer theory.

Although focussed on lattice polymer theory, Comets and Yoshida (2006) is likely state of the art for the research frontiers on dispersion problems for the case of lattice polymers. However, less actually appears to be available in the literature explicitly focussed on the case of tree polymers in terms of precise bounds and results (see remarks of the next section). Since the importance of tree polymers is precisely that of furnishing sharp results, part of the purpose of the present paper is to provide a complete and self-contained treatment of best possible bounds and results for the basic dispersion problem of tree polymer theory.

This leads to a number of additional interesting open mathematical problems for tree polymers from the perspective of Kahane's **T**-martingale theory, where most of the focus has heretofore primarily been on describing the fine scale structure of a.s. surviving cascades (weak disorder). In fact it motivates a number of entirely new questions for the continued development of **T**-martingale theory both in the case of weak and strong disorder types. For example, as will be seen, even in the analysis of weak disorder the approach of this paper involves *differentiation* of a certain class of **T**-martingales along lines introduced by Barral (2000). This naturally leads to a new notion of **signed** or **complex T**-martingales for which the authors know of no general theory. The strong disorder problems involve very new phenomena that escape direct application of existing theory.

In the next section the tree polymer model is introduced. Bolthausen's notion of **weak** and **strong disorder** environments, respectively, are precisely defined and some basic polymer problems are identified. Section 3 contains a brief summary of Kahane's **T**-martingale theory appropriate to this application, as well as an overview of the extension of Peyrière's mean size-bias probability to strong disorder environments introduced in Waymire and Williams (1994). As something of a warm-up, Section 4 opens with a simple result demonstrating that regardless of disorder type, the tree polymer paths are **non-ballistic** in the sense of an almost sure law of large number convergence to zero. This is followed by a discussion of a polymer diffusion problem and includes asymptotic polymer **path free energy** type calculations in both cases of weak and strong disorder. These are in contrast to the free energy calculations provided by Buffet, Patrick, and Pulé (1993). The latter are made simpler by restricting their considerations to normalization constants. Section 5 contains the complete proof of a.s. long-chain Gaussian fluctuations under the $n^{\frac{1}{2}}$ diffusive scaling within the *full range* of weak disorder, (i.e., a tree polymer path CLT). In section 6 *vector cascades* are introduced as a class of **T**-martingales for which one can obtain an alternative representation of path free energy under strong disorder. Related directions and open questions for extensions of **T**-martingale theory are briefly discussed in the final section 7.

2 Background and Notation

Let $T := \cup_{n=0}^{\infty}\{-1,1\}^n$ denote the set of **vertices** of the complete binary tree rooted at 0, with the convention that $\{-1,1\}^0 := \{0\}$. Equivalently, each vertex $v \neq 0$ defines a unique *edge* adjacent to this vertex on the unique connected path joining the vertex v to the root 0. Such an edge is unambiguously also denoted by v. The **tree path space** is defined by the Cartesian product $\partial T := \{-1,1\}^{\mathbf{N}}$, where each $s = (s_1, s_2, \ldots) \in \partial T$ defines a possible *polymer path*. It is convenient to denote the vertex (or edge) at the $j-$th level of the path s, read "s restricted to j", by $s|j := (s_1, \ldots, s_j)$, for each $j = 1, 2, \ldots$, and $s|0 := 0$. The same notation applies to a **finite path segment** $t = (t_1, \ldots, t_n) \in \{-1,1\}^n$ for $j \leq n$. In this case, $|t| = n$ denotes the **length** of the finite path segment. The **polymer path position** of $s = (s_1, s_2, \ldots) \in \partial T$ at the nth link is defined by the special notation

$(s)_0 = 0$, $(s)_n = \sum_{j=1}^{n} s_j$, for $n \geq 1$. The **normalized Haar measure** $\lambda(ds)$ on ∂T, regarded as a compact Abelian group under coordinatewise multiplication and the Cartesian product topology for the discrete topology on each factor $\{-1, 1\}$, defines the uniform distribution on polymer path space ∂T.

Next, the **environment** is defined by a collection $\{X(v) : v \in T\}$ of i.i.d. (strictly) positive random variables on a probability space $(\Omega, \mathcal{F}, P)$ indexed by the vertices (or edges) of T. Define a sequence of random probability measures $prob_n(ds, \omega) << \lambda(ds)$, $n \geq 1, \omega \in \Omega$, by the corresponding sequence of Radon–Nikodym derivatives

$$\frac{dprob_n}{d\lambda}(s, \omega) = Z_n^{-1}(\omega) \prod_{j=1}^{n} X(s|j), \quad n = 1, 2, \ldots,$$

where $Z_n(\omega)$ denotes the normalization constant (or **partition function**) given by

$$Z_n(\omega) := \int_{\partial T} \prod_{j=1}^{n} X(s|j) \lambda(ds) = \sum_{|t|=n} \prod_{j=1}^{n} X(t|j)(\omega) 2^{-n},$$

and the sum is over all finite path segments t of length n. In particular on the **finite dimensional cylinder sets** $\Delta_n(t) := \{s \in \partial T : s|n = t\}$, $t \in \{-1, 1\}^n$, $n \geq 1$ of the Borel $\sigma-$field of ∂T one has

$$prob_n(\Delta_n(t), \omega) = Z_n^{-1}(\omega) \prod_{j=1}^{n} X(t|j)(\omega) 2^{-n}.$$

The factor 2^{-n} cancels in the ratio, but is convenient to display as it makes the sequence $\{Z_n/(E_P X)^n : n \geq 1\}$ a positive martingale. Observe, also, that for each finite dimensional cylinder set $\Delta_n(t)$, one has sample pointwise on Ω

$$prob_{n+m}(\Delta_n(t)) = \frac{Z_n \sum_{|s|=m} \prod_{j=1}^{m} X(t * (s|j)) 2^{-m}}{Z_{n+m}} prob_n(\Delta_n(t)),$$

where $*$ denotes concatenation of word strings defining vertices.

Definition 1. *Given an environment $\{X(v) : v \in T\}$ of i.i.d. (strictly) positive random variables on a probability space $(\Omega, \mathcal{F}, P)$ indexed by the vertices (or edges) of T, the **tree polymer** is defined by the sequence $prob_n(dt)$, $n \geq 1$, of (random) probabilities defined on the Borel sigmafield of ∂T.*

Some special notation and assumptions

The explicit dependence of random variables on $\omega \in \Omega$ will generally be suppressed as per standard probability convention. Also, a number of different probabilities will appear throughout this paper, e.g., P, $prob_n$, $\mathcal{Q}$, etc., whose role in expected value computations will be indicated by a subscript to the expectation symbol E. It will be assumed throughout that there is a number $p > 1$ such that

$$E_P X^p < \infty. \tag{1}$$

This condition is easily satisfied by the following basic examples of polymer theories, namely (i) $X = e^{\beta Z}$, for standard normal Z, and (ii) $X = \begin{cases} a & \text{with probability } p \\ b & \text{with probability } q = 1-p, \end{cases}$ for some $a, b > 0$, $0 < p < 1$. Without loss of generality one may take $E_P X = 1$, since replacing X by $X/E_P X$ is canceled by the respective factors $(E_P X)^n$ of the new normalization constants. This normalization will also be assumed throughout, modifying the form of these examples accordingly.

In view of the martingale convergence theorem $Z_\infty = \lim_{n\to\infty} Z_n$ exists P-a.s. Also by Kolmogorov's zero-one law and sure positivity of the environmental weights, the event $[Z_\infty = 0]$ has P-probability zero or one. The environment $\{X(v) : v \in T\}$ is referred to as **weak disorder** if and only if P-a.s. $Z_\infty > 0$, otherwise the environment is that of **strong disorder**. In the case of weak disorder one has the existence of an a.s. unique **t**ree polymer limit probability $prob_\infty(dt)$ on ∂T defined by the a.s. weak limit of the tree polymer $prob_n(dt), n \geq 1$. In Waymire and Williams (1996) the existence of a unique weak limit probability was proven under strong disorder as a Dirac point mass concentrated on a random path $\tau \in \partial T$ with respect to the mean size-biasing change of P-measure described in the next section. Moreover the mean size-biasing change of measure and P are mutually singular under strong disorder.

Remark 1. Yuval Peres (personal communication) suggested that under strong disorder the set of limit points of $prob_n(dt), n \geq 1$, might a.s. consist of Dirac point masses on paths.

Remark 2. The sharp criticality condition for transitions between weak and strong disorder is known precisely for tree polymer models as a result of the seminal paper of Kahane and Peyrière' (1976). In addition, Bolthausen's weak/strong disorder criticality condition was improved by Birkner (2004) for the case of lattice polymers using a size-bias change of measure. Birkner's criticality condition indeed coincides with the sharp determination that one obtains using the Kahane and Peyrière (1976) theory for the case of tree polymers. This illustrates a benchmark role for tree polymer theory for evaluating the sharpness of lattice polymer methods mentioned at the outset.

In the context of tree polymers, the basic theory concerns the P-a.s. asymptotic behavior of segments of *random* polygonal paths $S \in \partial T$ of length n distributed, respectively, according to the sequence $prob_n(ds, \omega)$. For example, a.s. strong laws governing averages $\frac{(S)_n}{n}$, and a.s. limit distributions governing fluctuations $\frac{(S)_n - c_n}{a_n}$ for suitable centering c_n and scaling constants $a_n > 0$, as $n \to \infty$ are desired. While these are only a few of the problems of interest here, they do play a central role.

Remark 3. The L^2-martingale methods developed in Bolthausen (1989, 1991) for the lattice polymer do indeed provide the CLT for tree polymers with $c_n = 0, a_n = \sqrt{n}$, but in a *strict* subregion of weak disorder. Comets and Yoshida (2006) note that subsequent L^2-martingale methods developed for lattice polymers extend the range of weak disorder in sufficiently high dimensions d. Specifically, Albevario and Zhou (1996), Imbrie and Spencer (1988), Song and Zhou (1996), Birkner (2004), are noted in this regard. In these results, however, the asymptotic diffusion coefficient in $d+1$ dimensions is $\frac{1}{d}$. The extension of these results and methods to the context of tree polymers does not seem obvious, although it appears that they are presumed to hold.

The present paper will provide an explicit, self-contained and complete proof for the CLT problem in the case of tree polymers of weak disorder type based on differentiated cascades. The sharpness obtained for tree polymers suggests that corresponding methods might also prove useful for lattice polymers. It is also shown that the same diffusive scaling limit is not possible under strong disorder. More generally, another important motivation for this paper is to uncover the extent of applicability of existing **T**-martingale theory and identify new directions under strong disorder.

3 T-martingales and size-bias theory

For a complete metric space $(\mathbf{T}, d)$, Kahane's **T-martingale** refers to a sequence of non-negative random functions $Q_n, n = 1, 2, \ldots$ on $\mathbf{T}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ adapted to a filtration $\mathcal{F}_n, n = 1, 2 \ldots$, such that for each $t \in \mathbf{T}$, $Q_n(t)$ is a mean-one martingale with respect to this filtration. Given a Radon measure σ on the Borel sigmafield $\mathcal{B}(\mathbf{T})$, the **T**-martingale induces a sequence of random measures $Q_n\sigma(dt)$ defined by

$$\int_{\mathbf{T}} f(t) Q_n \sigma(dt) = \int_{\mathbf{T}} f(t) Q_n(t) \sigma(dt)$$

for all continuous bounded functions f on $\mathbf{T}$, i.e., $\frac{dQ_n\sigma}{d\sigma}(t) = Q_n(t), t \in \mathbf{T}$. As such, using martingale convergence theory and Kahane's **T**-martingale decomposition, e.g., see Kahane (1987b), Waymire and Williams (1994), one may obtain a (possibly degenerate) random measure $Q_n\sigma \Rightarrow \sigma_\infty$ as an a.s. vague limit.

For the case of tree polymers consider $\mathbf{T} = \partial T$ introduced in the previous section, with

$$Q_n(s) = \prod_{j=1}^{n} X(s|j), \quad s \in \partial T,$$

and for example, $\sigma = \lambda$, the Haar measure on ∂T. As noted earlier, in the context of tree polymers $X(v), v \in t$ is a strictly positive, mean one random variable. One may write

$$prob_n(ds) = \frac{Q_n\lambda(ds)}{Q_n\lambda(\partial T)}, \qquad n \geq 1.$$

and, in the case of weak disorder, one has

$$prob_\infty(ds) = \frac{\lambda_\infty(ds)}{\lambda_\infty(\partial T)}.$$

Peyrière's **mean size bias** was introduced to compute fine scale structure of surviving cascades, i.e., weak disorder in the context of polymers. The consideration of such transformations is naturally motivated by more basic elements of Cramèr-Chernoff exponential size-biasing in the computation of large deviation rates, e.g., see Bhattacharya and Waymire (2007). Specifically, since the product of i.i.d. mean one nondegenerate random variables along any one path is a.s. zero, the survival of cascades requires deviations from this average behavior made possible by the uncountably many paths of ∂T. Moreover, exponential size biasing of the logarithm of a random variable is precisely mean size biasing. We summarize here the basic framework developed in Waymire and Williams (1994), (1995), (1996) to use size biasing to determine the asymptotic total mass in cases of both weak and strong disorder.

By restricting the formulation to the sigmafields generated, respectively, by the first finitely many levels of the environment $\mathcal{F}_n := \sigma(X(v) : |v| \leq n)$, and the finite dimensional cylinder sets of tree paths $\mathcal{R}_n := \sigma(\Delta_n(t) : |t| = n)$, for $n \geq 1$, with the aid of the Kolmogorov consistency theorem, one may define a joint probability $\mathcal{Q}(d\omega \times dt)$ (on $\Omega \times \partial T$) of the environment and paths that, for a given path s, size biases the environment along this path. Namely, one has

$$\mathcal{Q}(d\omega \times ds) = \prod_{j=1}^{n} X(s|j)(\omega)P(d\omega)\lambda(ds) = P_s(d\omega)\lambda(ds),$$

where

$$P_s << P \quad \text{on} \quad \mathcal{F}_{\rm n} = \sigma(\rm X(v) : |v| \leq n).$$

In other words, the measures $\prod_{j=1}^{n} X(s|j)(\omega)P(d\omega)\lambda(ds), n \geq 1$, provide a consistent specification of the finite dimensional distributions of $(\{X(v) : v \in T\}, S)$ under $\mathcal{Q}(d\omega \times ds)$ on $\mathcal{F} \otimes \mathcal{B}$. Accordingly, under $\mathcal{Q}(d\omega \times ds)$, for a given path $S = s$, the environment variable $X(v)$ has distribution $P \circ X^{-1}(dx)$ if v is not on s, while it is $xP \circ X^{-1}(dx)$ if v is along the path s.

Letting $\pi_\Omega\ \pi_{\partial T}$ denote the coordinate projection maps of $\Omega \times \partial T$ onto Ω and ∂T, respectively, one obtains by integrating out the coordinates that

$$(i)\ \ \mathcal{Q} \circ \pi_\Omega^{-1}(d\omega) = Z_n(\omega)P(d\omega), \qquad (ii)\ \ \mathcal{Q} \circ \pi_{\partial T}^{-1}(ds) = \lambda(ds). \qquad (3)$$

From here one readily obtains the following variant on *Bayes formula*

$$\mathcal{Q}(d\omega \times ds) = prob_n(ds,\omega)\mathcal{Q}\circ\pi_\Omega^{-1}(d\omega). \tag{4}$$

In particular, the polymer path distribution $prob_n(ds,\omega)$ is the conditional path probability given the environment.

Next one has the *Lebesgue decomposition*

$$\mathcal{Q}(d\omega \times ds) = Q_\infty\lambda(ds,\omega)P(d\omega) + \mathbf{1}[Q_\infty\lambda(\partial T,\omega)=\infty]p_\infty(ds,\omega)\mathcal{Q}\circ\pi_\Omega^{-1}(d\omega), \tag{5}$$

where $p_\infty(ds,\omega)$ denotes the $\mathcal{Q}\circ\pi_\Omega^{-1}$-a.s. weak limit of $p_n(ds)$ as $n\to\infty$. The structure of $p_\infty(ds,\omega)$ in the case of strong disorder is described in Proposition 2 below. Accordingly, with regard to weak and strong disorder, the event $[Q_\infty\lambda(\partial T)=0]$ is a zero-one event under P if and only if $[Q_\infty\lambda(\partial T)=\infty]$ is a zero-one event under $\mathcal{Q}\circ\pi_\Omega^{-1}(d\omega)$.

Next we record the *(weighted) first departure bounds* developed in Waymire and Williams (1996) for the special case of the product probabilities $\sigma = \mu\times\mu\times\cdots$ of a (generic) Bernoulli probability μ on $\{-1,1\}$ that will naturally appear in forthcomming tree polymer applications. Namely, for an arbitrary path $s\in\partial T$, and positive constants C_n, one has

$$\prod_{j=1}^n X(s|j)\mu(+)^{\#(s|n)}\mu(-)^{\#(s|n)} \le Q_n\sigma(\partial T) \le \prod_{j=1}^n X(s|j)\mu(+)^{\#^+(s|n)}\mu(-)^{\#^-(s|n)} + C_nA_n, \tag{6}$$

where $C_n>0$ and $A_n, n\ge 1$ is a positive submartingale (dependent on the choice of C_n). The symbols $\#^\pm(s|n)$ count the respective number of ±1 coordinates of the path segment $s|n$, and $\mu(\pm)=\mu(\{\pm1\})$, respectively. The lower bound is obvious since a sum of positive terms is larger than any single term. The upper bound is obtained by splitting off the term corresponding to the product along the $s-$path and decomposing the remaining sum with respect to the level of first departure from the $s-$path.

This summarizes the essential elements of the theory which will be needed for this paper.

4 Asymptotic polymer path free energy type calculations for weak and strong disorder

In addressing the asymptotic structure of tree polymers without regard to disorder type one is forced to consider weak limits; i.e., limits with respect to the sequence $prob_n(dt), n\ge 1$. The following simple lemma is somewhat surprising on first glance in view of the random normalization.

Lemma 1. *On $\mathcal{R}_n$ one has*

$$E_P prob_n(B) = \lambda(B) = E_P Q_n\lambda(B), \quad B\in\mathcal{R}_n.$$

Proof. Simply observe that $E_P prob_n(\Delta_n(t)) = 2^{-n}$ since the expression is independent of $t \in \partial T$ and sums to one.

As an application of this lemma one can readily obtain an expression of the *n*onballistic character of polymers regardless of disorder type.

Proposition 1. *Regardless of the disorder strength one has*

$$\lim_{n\to\infty} E_{prob_n} |\frac{(S)_n}{n}| = 0 \quad P - a.s.$$

Proof. Let $A_n = E_{prob_n}|\frac{(S)_n}{n}|$. Then for $h > 1$, applying Jensen's inequality to the integral with respect to $prob_n(ds)$, one has

$$E_P A_n^h = n^{-h} E_P \left(\int_{\partial T} |(s)_n| prob_n(ds) \right)^h \leq n^{-h} \int_{\partial T} |(s)_n|^h E_P prob_n(ds) \leq C n^{-\frac{h}{2}}$$

with $C > 0$. Now take $h = 4$ and apply Borel-Cantelli to obtain the assertion.

The result quoted in the previous section that identifies $prob_\infty(ds)$ in the case of strong disorder as concentrated on a single random path $\mathcal{Q} \circ \pi_\Omega^{-1}(d\omega)-$a.s. is repeated here for ease of reference and to correct some typographical errors in the proof in WW96B.

Proposition 2 (WW96B). *In the case of strong disorder there is a random path $\tau = \tau(\omega) \in \partial T, \omega \in \Omega$, such that $\mathcal{Q} \circ \pi_\Omega^{-1}(d\omega)-a.s.$ as $n \to \infty$,*

$$prob_n(ds) \Rightarrow \delta_\tau(ds).$$

Proof. Fix a path s. If, for example $s_1 = +1$, then the total mass on the "left side" of the tree, $Z_n(-) = \sum_{|t|=n, t_1=-1} \prod_{j=1}^n X(t|j) 2^{-(n-1)}$ is a positive martingale under P_s since the environment off the path s is i.i.d. distributed under P. In particular, therefore, P_s-a.s. one has

$$Z_\infty(-) = \lim_{n\to\infty} Z_n(-) < \infty.$$

A similar observation holds if $s_1 = -1$, and so on down the tree off the path s. But under strong disorder, for any path s, since $Z_\infty = Q_\infty \lambda(\partial T) = 0 \;\; P-$a.s., from the Lebesgue decomposition one observes that

$$P_s(Z_\infty = \infty) = 1.$$

Let $\omega \in [Z_\infty = \infty]$. Then, removing an event of $\mathcal{Q} \circ \pi_\Omega^{-1}$-probability zero if necessary, one has either $Z_\infty(+, \omega) = \infty$ or $Z_\infty(-, \omega) = \infty$, but not both. Define $\tau_1(\omega) = \pm 1$ according to $Z_\infty(\pm, \omega) = \infty$. Now iterate this procedure down the tree accordingly.

For the a.s. distributional limits of interest in the next two sections, it will be convenient to have calculations of the a.s. asymptotic behavior of **polymer path free energies** (or *cumulant generating functions*) of the form $F(r) = \lim_{n\to\infty} \frac{\ln M_n(r)}{n}$, where

$$M_n(r) = E_{prob_n} e^{r(S)_n}.$$

Remark 4. In Buffet, Patrick, and Pulé (1993) the authors consider a different type of free energy density calculations which, in the present notation, may be defined for environments $X = e^{-\beta V}$, (for a particular class of real-valued random variables V), as

$$\psi = \lim_{n\to\infty} \frac{\ln Z_n}{n},$$

where Z_n is the corresponding normalizing constant (partition function). Such considerations will follow as a special case of path free energy results presented here.

Lemma 2. *Let*

$$p_r(\pm 1) = \frac{e^{\pm r}}{e^r + e^{-r}}, \quad \lambda_r = p_r \times p_r \times \cdots \times p_r \times \cdots.$$

Then

$$M_n(r) = \cosh^n(r) \frac{Q_n \lambda_r(\partial T)}{Q_n \lambda_0(\partial T)}, \quad -\infty < r < \infty.$$

Proof. One has

$$\begin{aligned} M_n(r) &= Z_n^{-1} \sum_{|s|=n} \prod_{j=1}^n e^{rs_j} \prod_{j=1}^n X(s|j) 2^{-n} \\ &= \cosh^n r Z_n^{-1} \sum_{|s|=n} \prod_{j=1}^n p_r(s_j) \prod_{j=1}^n X(s|j) \\ &= \cosh^n r \frac{\sum_{|s|=n} \prod_{j=1}^n X(s|j) \prod_{j=1}^n p_r(s_j)}{\sum_{|s|=n} \prod_{j=1}^n X(s|j) 2^{-n}}. \end{aligned} \tag{7}$$

This completes the proof.

The following formula is well-known by various methods starting with Borel normal numbers and its extensions by Eggleston (1949), Billingsley (1960), Kifer (1996), Fan (1994), and Peyrière (1977). We write *supp*σ for the maximal Borel support of a probability σ on ∂T. That is *supp*$\sigma = \inf\{dim(A) : \sigma(A^c) = 0\}$, where dim$A$ denotes the Hausdorff dimension of Borel $A \subseteq \partial T$ (for the metric $\rho(s,t) = 2^{-|s\wedge t|}, s,t \in \partial T$, where $s \wedge t$ denotes the common part of the paths s, t emanating from the root, until first departure). With this notation and terminology one has,

$$dim(supp\lambda_r) = h_2(r) := -\frac{e^r}{e^r+e^{-r}}\log_2\left(\frac{e^r}{e^r+e^{-r}}\right) - \frac{e^{-r}}{e^r+e^{-r}}\log_2\left(\frac{e^{-r}}{e^r+e^{-r}}\right). \tag{8}$$

We also refer to $h_2(r)$ as the *base 2-entropy* of λ_r. In particular, note that the Haar measure (uniform distribution) λ_0 has full support of dimension one, i.e., *maximal entropy* among λ_r, $-\infty < r < \infty$.

The following is a special case of more general theorems of Kahane (1987a) on conditions for survival of multiplicative cascades with respect to initial measures σ on ∂T using potential theoretic/capacity methods. In the case of product measures, such as λ_r this also follows from the weighted size-bias theory developed in Waymire and Williams (1996). It may also be obtained from necessary and sufficient conditions obtained by Fan (2002) for Markov measures. In essence, the support must be sufficiently large relative to the variability in the environment for the cascade to survive. In the case of Haar measure λ_0, this may be equivalently interpreted as the condition that the branching number 2 must be large enough relative to variability of the environment. Namely,

Proposition 3. *For arbitrary $r \in \mathbf{R}$ one has*

$$Q_\infty\lambda_r(\partial T) > 0 \quad \text{a.s.}$$

if and only if

$$E_P X \log_2 X < h_2(r).$$

Proof. The proof follows precisely the lines of Waymire and Williams (1994), using the weighted first departure bounds. For necessity, suppose that $E_P X \log_2 X \geq h_2(r)$. Then, for any fixed path $s \in supp(\lambda_r)$, one has using the lower bound

$$\begin{aligned} Q_n\lambda_r(\partial T) &\geq \prod_{j=1}^{n} X(s|j) p_r^{\#^+(s|n)}(+) p_r^{\#^-(s|n)}(-) \\ &= \exp\{n\left(\frac{1}{n}\sum_{j=1}^{n}\ln X(s|j) + \frac{\#^+(s|n)}{n}\ln p_r(+) + \frac{\#^-(s|n)}{n}\ln p_r(-)\right)\}. \end{aligned}$$

By two applications of the strong law of large numbers, one has, respectively, that P_s–a.s. $\frac{1}{n}\sum_{j=1}^{n}\ln X(s|j) \to E_P X \ln X$, and λ_r– a.e. $\frac{\#^\pm(s|n)}{n} \to p_r^\pm$ as $n \to \infty$. It follows from this that

$$\int_{\partial T}\int_{\Omega} \mathbf{1}[Q_\infty(\partial T) = \infty] P_s(d\omega)\lambda_r(ds) = 1$$

in the case $E_P X \log_2 X > h_2(r)$. The same can be seen to hold when $E_P X \log_2 X = h_2(r)$ using the Chung-Fuchs theorem in place of the strong law of large numbers. The converse is proved similarly using the upper first departure bound.

Remark 5. In the case of Haar measure $\lambda = \lambda_0$, $h_2(0) = \ln 2$ and Proposition 3 provides the usual condition on the variability in the environment with respect to the branching number for weak and strong disorder.

Corollary 1. *Under weak disorder one has P-a.s. that there is a $\delta > 0$ such that*

$$F(r) = \lim \frac{\ln M_n(r)}{n} = \ln\cosh(r) \quad |r| \leq \delta.$$

Proof. Since weak disorder is equivalent to $E_P X \log_2 X < h_2(0)$, and $h_2(0)$ is maximal entropy, using continuity of $h_2(r)$, there is a $\delta > 0$ such that $E_P X \log_2 X < h_2(r)$ for $|r| \leq \delta$. The result follows immediately from Proposition 3 taking logarithms in Lemma 2.

Remark 6. Observe that in the case of simple symmetric random walk paths obtained by taking deterministic $X \equiv 1$, one has the sure identity

$$\frac{\ln M_n(r)}{n} \equiv \ln\cosh(r), \quad n = 1, 2, \ldots.$$

Moreover

$$\cosh^n(\frac{r}{\sqrt{n}}) \sim \left(1 + \frac{r^2}{2n} + o(1)\right)^n \sim e^{\frac{r^2}{2}} \quad \text{as n} \to \infty.$$

So formally, at least, one expects the diffusive (CLT) limit to hold almost surely from Corollary 1.

The computation of the path free energy under strong disorder is a little more delicate than the case of Corollary 1. We will make a size-bias calculation for an upper bound on lim sup. However, the lower bound on lim inf obtained by the corresponding approach is too small. Nonetheless we will see that the lim sup bound is indeed the asserted a.s. limit.

Proposition 4. *Under strong disorder there is a $\delta > 0$ such that*

$$F(r) = \lim \frac{\ln M_n(r)}{n}$$
$$= \ln\cosh(r) + \frac{\ln E_P X^{h(r)} + \ln\left(p_r^{h(r)}(+) + p_r^{h(r)}(-)\right)}{h(r)} - \frac{\ln E_P X^{h(0)} - (h(0) - 1)\ln 2}{h(0)},$$

where $h = h(r)$ is uniquely determined positive solution to

$$E_P\left\{\frac{X^h}{E_P X^h} \ln \frac{X^h}{E_P X^h}\right\} = \epsilon(\overline{p}_{r,h}(+), \overline{p}_{r,h}(-)),$$

for

$$\overline{p}_{r,h}(\pm) := \frac{p_r^h(\pm)}{p_r^h(+) + p_r^h(-)}$$

and $\epsilon(a, b) = -a\ln a - b\ln b$.

Proof. We begin by using size-biasing to compute an upper bound on $limsup_{n\to\infty}\frac{\ln Q_n\lambda_r(\partial T)}{n}$. Fix $c > 0$, $0 < h < 1$. The size-bias change of measure in this context may be obtained by the modification denoted

$$\mathcal{Q}^{(r)}(d\omega \times ds) = P_s(d\omega)\lambda_r(ds).$$

In particular, on $\mathcal{F}_n$

$$\mathcal{Q}^{(r)}\circ\pi_{\Omega}^{-1}(d\omega) = \int_{\partial T} P_s(d\omega)\lambda_r(ds) = \sum_{t\in\{-1,1\}^n}\int_{\Delta_n(t)}\prod_{j=1}^{n} X(t|j)P(d\omega)\lambda_r(dt) = m_n(r)P(d\omega),$$

where

$$m_n(r) := \frac{Z_n M_n(r)}{\cosh^n(r)}.$$

Now,

$$\begin{aligned}
&P(Q_n\lambda_r(\partial T) > e^{nc})\\
&= E_P\mathbf{1}[Q_n\lambda_r(\partial T) > e^{nc}]\\
&= E_{\mathcal{Q}^{(r)}\circ\pi_{\Omega}^{-1}} m_n(r)^{-1}\mathbf{1}[m_n(r) > e^{nc}]\\
&\le \int_{\partial T}\int_{\Omega}\frac{m_n^h(r)e^{-nch}}{m_n(r)}P_s(d\omega)\lambda_r(ds)\\
&\le e^{-nhc}\int_{\partial T}\int_{\Omega}\frac{1}{\prod_{j=1}^n X^{1-h}(s|j)p_r^{1-h}(s_j)}P_s(d\omega)\lambda_r(ds)\\
&= e^{-nhc}\int_{\partial T}\int_{\Omega}\prod_{j=1}^{n} X^h(s|j)p_r(s_j)^{h-1}\frac{1}{\prod_{j=1}^n X(s|j)}P_s(d\omega)\lambda_r(ds)\\
&= e^{-nhc}(E_P X^h)^n\int_{\partial T}\prod_{j=1}^{n} p_r^{h-1}(s_j)\lambda_r(ds) = e^{-nhc}(E_P X^h)^n\left(\int_{\partial T} p_r^{h-1}(s_1)\lambda_r(ds)\right)^n\\
&= \exp\{-n\left[hc - (\ln E_P X^h + \ln(p_r^h(+) + p_r^h(-)))\right]\}.
\end{aligned}$$

Thus, the probability is summable for

$$c > \inf_{0<h<1}\frac{\ln E_P X^h + \ln(p_r^h(+) + p_r^h(-))}{h}.$$

Using Borel-Cantelli one therefore obtains P-a.s. that

$$\limsup_{n\to\infty}\frac{\ln Q_n\lambda_r(\partial T)}{n} \le \inf_{0<h<1}\frac{\ln E_P X^h + \ln(p_r^h(+) + p_r^h(-))}{h}.$$

Next we verify that this upper bound on the limsup is also a lower bound on the liminf, and therefore is the desired limit P-almost surely. Define for fixed r,

$$Z_n(r,h) := \sum_{|s|=n} \prod_{j=1}^{n} X^h(s|j) p_r^h(s_j), \quad h \in \mathbf{R}.$$

Then

$$Q_n \lambda_r(\partial T) = Z_n(r,1).$$

Also note that

$$E_P Z_n(r,h) = (E_P X^h)^n \left(p_r^h(+) + p_r^h(-)\right)^n.$$

Viewing X^h as a new polymer environment, and normalizing $p_r^h(\pm)$ to a probability distribution given by

$$\overline{p}_{r,h}(\pm) = \frac{p_r^h(\pm)}{p_r^h(+) + p_r^h(-)},$$

one sees from Proposition 3 that for each fixed r there is a unique $h(r)$ defined by

$$E_P \frac{X^h}{E_P X^h} \ln \frac{X^h}{E_P X^h} = \epsilon(r,h),$$

where $\epsilon(r,h) = -\overline{p}_{r,h}(+) \ln \overline{p}_{r,h}(+) - \overline{p}_{r,h}(-) \ln \overline{p}_{r,h}(-)$, such that

$$\lim_{n\to\infty} \frac{Z_n(r,h)}{E_P Z_n(r,h)} > 0 \quad P - a.s.$$

if and only if $h < h(r)$. Thus, for $h < h(r)$, one has

$$\lim_{n\to\infty} \frac{\ln Z_n(r,h)}{n} = \ln E_P X^h + \ln \left(p_r^h(+) + p_r^h(-)\right).$$

The uniqueness of $h = h(r)$ follows by checking that for fixed r, $h \to E_P \frac{X^h}{E_P X^h} \ln \frac{X^h}{E_P X^h} - \epsilon(r,h)$, is monotone increasing on $0 < h < 1$. Define

$$g(r,h) := \ln E_P X^h + \ln \left(p_r^h(+) + p_r^h(-)\right).$$

Now, for $\epsilon > 0$, rewrite a bit, and apply Jensen's inequality to obtain

$$\begin{aligned}
\frac{Z_n(r,1)}{Z_n(r,h)} &= Z_n(r,h)^{-1} \sum_{|s|=n} \prod_{j=1}^{n} X(s|j) p_r(s_j) \\
&= \sum_{|s|=n} \prod_{j=1}^{n} X^{1-h}(s|j) p_r^{1-h}(s_j) Z_n(r,h)^{-1} \prod_{j=1}^{n} X^h(s|j) p_r^h(s_j) \\
&= \sum_{|s|=n} \left(\prod_{j=1}^{n} X^{\frac{1-h}{1+\epsilon}}(s|j) p_r^{\frac{1-h}{1+\epsilon}}(s_j) \right)^{1+\epsilon} Z_n(r,h)^{-1} \prod_{j=1}^{n} X^h(s|j) p_r^h(s_j) \\
&\geq \frac{\left(\sum_{|s|=n} \prod_{j=1}^{n} X^{\frac{1-h}{1+\epsilon}+h}(s|j) p_r^{\frac{1-h}{1+\epsilon}+h}(s_j) \right)^{1+\epsilon}}{Z_n(r,h)^{1+\epsilon}}.
\end{aligned}$$

Thus,

$$Q_n\lambda_r(\partial T) \equiv Z_n(r,1) \geq \frac{Z_n^{1+\epsilon}(r, h + \frac{1-h}{1+\epsilon})}{Z_n^{\epsilon}(r,h)}.$$

In particular, therefore,

$$\begin{aligned}\frac{\ln Z_n(r,1)}{n} &\geq (1+\epsilon)\frac{\ln Z_n(r, \frac{1+\epsilon h}{1+\epsilon})}{n} - \epsilon\frac{\ln Z_n(r,h)}{n} \\ &= \frac{\ln Z_n(r, \frac{1+\epsilon h}{1+\epsilon})}{n} + \epsilon\left[\frac{\ln Z_n(r, \frac{1+\epsilon h}{1+\epsilon})}{n} - \frac{\ln Z_n(r,h)}{n}\right].\end{aligned}$$

Now, $\frac{1+\epsilon h}{1+\epsilon} < h(r)$ for $\epsilon > \frac{1-h(r)}{h(r)-r} > 0$. Thus, taking liminf as $n \to \infty$, followed by letting $\epsilon \downarrow \frac{1-h(r)}{h(r)-h}$, yields

$$\liminf_{n\to\infty} \frac{\ln Z_n(r,1)}{n} \geq g(r,h(r)) + \frac{1-h(r)}{h(r)-h}\left[g(r,h(r)) - g(r,h)\right].$$

Finally, let $h \uparrow h(r)$ to obtain,

$$\liminf_{n\to\infty} \frac{\ln Z_n(r,1)}{n} \geq g(r,h(r)) + (1-h(r))\frac{\partial g}{\partial h}(r,h(r)).$$

With a bit of tedious algebra one may check that the size-bias bound on the limsup coincides with this lower bound on the liminf and, therefore, is the a.s. limit. The limit asserted by the proposition now follows.

5 Diffusive limits under full range of weak disorder

Taking the deterministic environment $X \equiv 1$ for which the tree polymer paths are then distributed as simple symmetric random walks, one clearly has

$$\frac{(S)_n}{\sqrt{n}} \Rightarrow Z \quad n \to \infty,$$

where Z has the standard normal law. The objective here is to show that this law a.s. persists throughout the entire range of weak disorder.

Remark 7. As remarked earlier, from the a.s. calculation $F(r) = \ln\cosh(r)$, one expects to a diffusive scaling limit to hold. A theorem of Ellis (1985) is known to lead from asymptotic calculations of the form $F(r) = \lim_n \ln M_n(r)/n$ under sufficient convexity conditions of such functions and their derivatives; e.g., see Cox and Griffeath (1985), Maxwell (1998) for indications of successful applications to certain particle systems and to cetain asymptotic enumerations, respectively. However, in the present case, even taking $X \equiv 1$ one observes changes in sign in the first derivative, e.g.,

$$(\ln\cosh(r))''' = -8\frac{e^r - e^{-r}}{(e^r + e^{-r})^3}.$$

The following lemma follows from straightforward calculations that are left to the reader to verify.

Lemma 3. *Let $\delta > 0$ be arbitrary. (i) $m_n(r) := \frac{M_n(r)}{\cosh^n(r)}, -\delta \le r \le \delta$, is a continuously differentiable $\boldsymbol{T}$-martingale on $\boldsymbol{T} = [-\delta, \delta]$ with the usual euclidean metric. Also the corresponding derived processes (ii) $m_n'(r) \equiv \frac{dm_n(r)}{dr}, -\delta \le r \le \delta$, is a (signed) $\boldsymbol{T}$-martingale. Moreover,*

$$m_n'(r) = \sum_{j=1}^{n} m_{n,j}(r) = \frac{1}{\cosh^n(r)}\sum_{j=1}^{n}\int_{\partial T}\{s_j - \tanh(r)\}e^{r(s)_n}Q_n\lambda(ds),$$

where $m_{n,j}(r), 1 \le j \le n$, are defined by the indicated terms of the second sum.

Remark 8. As noted earlier, this lemma illustrates a natural role for the extended notions of signed (or more generally complex) **T**-martingales, as well as **T**-martingale difference sequences. The following lemma makes explicit use of the assumption (1).

To take advantage of the symmetries of the binary tree and environment, we say a permutation (i.e., bijection) $\pi : T \to T$ of $T := \cup_{n=0}^{\infty}\{-1,1\}^n$ is **lattice preserving** if for each $v \in T$, both (i) $|\pi(v)| = |v|$, and (ii) $\pi(v|j) = (\pi(v)|j)$, for $j \le |v|$. Let $\mathcal{P}_{\le n}$ denote the collection of lattice preserving permutations which also satisfy $\pi(u * v) = \pi(u) * v$, if $|u| = n$, for $u, v \in T$, where $*$ is concatenation of the two sequences. Now, for $A \in \mathcal{F} = \sigma(X(v) : v \in T)$, say $A = [X(v_1) \in B_1, \dots, X(v_k) \in B_k]$, write $\pi(A) = [X(\pi(v_1)) \in B_1, \dots, X(\pi(v_k)) \in B_k]$, $B_i \in \mathcal{B}(0,\infty)$. Define

$$\mathcal{S}_n := \{A \in \mathcal{F} : A = \pi(A)\ \forall\ \pi \in \mathcal{P}_n\}.$$

Lemma 4. *Under weak disorder, equivalently $E_P X \ln X < ln2$, there is a number $1 < q < 2$ and a positive number δ such that*

$$\lim_{n\to\infty}\sum_{j=1}^{n}\sup_{|r|\le\delta}||m_{n,j}(r)||_{L^q(\Omega,\mathcal{F},P)} < \infty.$$

Proof. For $1 < q < 2$, $\frac{q}{2} < 1$. Note that

$$||m_{n,j}(r)||_{L^q}^q = E_P|m_{n,j}(r)|^q = E_P\left(|m_{n,j}(r)|^2\right)^{\frac{q}{2}} \le E_P\left(E_P\{|m_{n,j}(r)|^2|\mathcal{S}_n\}\right)^{\frac{q}{2}}.$$

Here $E_P\{|m_{n,j}(r)|^2|\mathcal{S}_n\}$ is the essentially unique positive $\mathcal{S}_n$-measurable random variable guaranteed by the Radon-Nikodym theorem. However, $|m_{n,j}(r)|^2$ need not be integrable (with respect to P), so that the usual L^1-expectation needs to be replaced by the L^+-version. Let $\mathcal{P}_n$ denote the collection of permutations on $T_n := \cup_{k=0}^n \{-1,1\}^k$ "depending on at most the first n levels", i.e., $\pi \in \mathcal{P}_n$ if and only if there is a $\hat\pi \in \mathcal{P}_{\le n}$ such that $\pi = \hat\pi|_{T_n}$. With this notation one may compute

$$E_P\{|m_{n,j}(r)|^2|\mathcal{S}_n\} = \frac{1}{\cosh^{2n} r}\frac{1}{\#\mathcal{P}_n^2}\int_{\partial T}\int_{\partial T}\sum_{\pi\in\mathcal{P}_n}\{(\pi(s)(j)-\tanh(r))$$
$$\times\left((\pi(t)(j)-\tanh(r)\right)e^{r(\pi(t))_n}e^{r(\pi(s))_n}\}\left(\sum_{\gamma\in\mathcal{P}_n}\prod_{i=0}^n X(\gamma(s|i))X(\gamma(t|i))\right)\lambda(ds)\lambda(dt).$$

Moreover,

$$\frac{1}{\cosh^{2n} r}\frac{1}{\#\mathcal{P}_n}\sum_{\pi\in\mathcal{P}_n}\{(\pi(s)(j)-\tanh(r))\left((\pi(t)(j)-\tanh(r)\right)\times e^{r(\pi(t))_n}e^{r(\pi(s))_n}\}$$
$$=\begin{cases}0 \quad \text{if } j>|s\wedge t| \\ \frac{1}{2^j\cosh^{2j}(r)}\sum_{|s|=j}\{(s_j-\tanh(r))^2 e^{2r(s)_j} \quad \text{if } j\le |s\wedge t|\end{cases}$$
$$=\begin{cases}0 \quad \text{if } j>|s\wedge t| \\ \frac{(1-\tanh r)^2e^2+(-1-\tanh r)^2e^{-2}}{2\cosh^2(r)}\frac{\cosh^{j-1}(2r)}{\cosh^{2j-2}(r)} \quad \text{if } j\le |s\wedge t|.\end{cases}$$

Thus, one may write

$$E_P\{|m_{n,j}(r)|^2|\mathcal{S}_n\} = \mu_j(r)\int_{\partial T}\int_{\partial T}\mathbf{1}[|s\wedge t|\ge j]$$
$$\times\frac{1}{\#\mathcal{P}_n}\left(\sum_{\pi\in\mathcal{S}_n}\prod_{i=0}^n X(\pi(s|i))X(\pi(t|i))\right)\lambda(ds)\lambda(dt)$$
$$=\mu_j(r)\sum_{k=j}^n\sum_{|s|=k}2^{-2k}\prod_{i=0}^k X^2(s|k)\int_{\partial T}\prod_{i=0}^{n-k-1}X(s*(1)*t|i)\lambda(dt)$$
$$\times\int_{\partial T}\prod_{i=0}^{n-k-1}X(s*(-1)*t|i)\lambda(dt),$$

where

$$\mu_j(r) = \frac{(1-\tanh r)^2e^2+(-1-\tanh r)^2e^{-2}}{2\cosh^2(r)}\frac{\cosh^{j-1}(2r)}{\cosh^{2j-2}(r)},$$

and one makes the convention that

$$\int_{\partial T}\prod_{i=0}^{-1}X(s*(1)*t|i)\lambda(dt)\int_{\partial T}\prod_{i=0}^{-1}X(s*(-1)*t|i)\lambda(dt)\equiv 1.$$

Recall that for the models considered here there is a $p > 1$ such that $E_P X^p < \infty$. Under weak disorder, therefore, there is a $1 < \hat{q} < 2$ such that

$$\frac{E_P X^q}{2^{q-1}} < 1 \quad \text{forany } 1 < \text{q} < \hat{\text{q}}.$$

So, for $q \in (1, \hat{q})$, it follows that

$$\begin{aligned}
&||m_{n,j}(r)||_q^q \\
&\leq E_P\{\mu_j(r)\sum_{k=j}^{n}\sum_{|s|=k} 2^{-2k}\prod_{i=0}^{k} X^2(s|k) \\
&\times \int_{\partial T}\prod_{i=0}^{n-k-1} X(s*(1)*t|i)\lambda(dt)\int_{\partial T}\prod_{i=0}^{n-k-1} X(s*(-1)*t|i)\lambda(dt)\}^{\frac{q}{2}} \\
&\leq E_P\{\mu_j^{\frac{q}{2}}(r)\sum_{k=j}^{n}\sum_{|s|=k} 2^{-2k}\prod_{i=0}^{k} X^q(s|k) \\
&\times \left(\int_{\partial T}\prod_{i=0}^{n-k-1} X(s*(1)*t|i)\lambda(dt)\int_{\partial T}\prod_{i=0}^{n-k-1} X(s*(-1)*t|i)\lambda(dt)\right)^{\frac{q}{2}}\} \\
&\leq \mu_j^{\frac{q}{2}}(r)\sum_{k=j}^{n}\frac{(E_P X^q)^k}{2^{(q-1)k}} \\
&\leq \mu_j^{\frac{q}{2}}(r)\frac{(E_P X^q)^j}{2^{(q-1)j}}(1-\frac{E_P X^q}{2^{q-1}})^{-\frac{1}{q}}.
\end{aligned}$$

Next choose $\delta > 0$ sufficiently small that for $|r| \leq \delta$

$$\frac{\cosh(2r)}{\cosh^2(r)}\frac{E_P X^q}{2^{q-1}} < \frac{1}{2}(\frac{E_P X^q}{2^{q-1}} + 1).$$

Then, letting $C = \max_{|r|\leq\delta}\sqrt{\frac{(1-\tanh r)^2 e^2 + (-1-\tanh r)^2 e^{-2}}{2\cosh^2(r)}}$, it follows that

$$\lim_{n\to\infty}\sum_{j=1}^{n}\sup_{|r|\leq\delta}||m_{n,j}(r)||_{L^q(\Omega,\mathcal{F},P)} \leq C(1-\frac{E_P X^q}{2^{q-1}})^{-\frac{1}{q}}\sum_{j=1}^{\infty}\{\frac{1}{2}(\frac{E_P X^q}{2^{q-1}}+1)\}^j < \infty$$

as asserted.

We conclude this section with a main result of this paper for tree polymers under the full range of weak disorder.

Theorem 1. *Assume weak disorder. Then P-a.s. there is a $\delta > 0$ and an absolutely continuous random function G on $-\delta \leq r \leq \delta$, such that uniformly on $|r| \leq \delta$ one has*

$$\lim_{n\to\infty} m_n'(r) \to G(r), \quad -\delta \le r \le \delta.$$

In particular, P-a.s.

$$\frac{(S)_n}{\sqrt{n}} \Rightarrow Z$$

where Z has a standard normal distribution.

Proof. From the lemma there are numbers $\delta > 0$ and $1 < q < 2$ such that

$$M := \lim_{n\to\infty} \sum_{j=1}^{n} \sup_{|r|\le\delta} ||m_{n,j}(r)||_{L^q(\Omega,\mathcal{F},P)} < \infty.$$

Then

$$\begin{aligned}\left(E_P \int_{-\delta}^{\delta} |m_n'(r)|^q dr\right)^{\frac{1}{q}} &\le \sum_{j=1}^{n} \left(E_P \int_{-\delta}^{\delta} |m_{n,j}(r)|^q dr\right)^{\frac{1}{q}} \\ &\le \sum_{j=1}^{n} \left(2\delta \sup_{|r|\le\delta} E_P |m_{n,j}(r)|^q\right)^{\frac{1}{q}} \\ &\le (2\delta)^{\frac{1}{q}} M.\end{aligned}$$

Thus $m_n'(r)$ converges $P-$a.s. and for almost every $r \in [-\delta, \delta]$, to some $G(r)$. In fact, with $C_q := (\frac{q}{q-1})^q$, one has by the L^q-maximal inequality that

$$\begin{aligned}\int_{-\delta}^{\delta} E_P \sup_{n\le N} |m_n'(r)|^q dr &\le C_q \int_{-\delta}^{\delta} E_P |m_N'(r)|^q dr \\ &\le 2\delta C_q M^q.\end{aligned}$$

Thus, P-a.s. $m_n'(r) \to G(r)$ in $L^q([-\delta,\delta], dr)$. In particular, $m_\infty(r) = \lim_{n\to\infty} m_n(r)$, $-\delta \le r \le \delta$ is uniform and $m_\infty(r)$ is absolutely continuous with derivative $G(r)$. Note that $m_\infty(0) = 1$ since $m_n(0) \equiv 1$ for each n. Since $m_\infty(r)$ is P-a.s. continuous in a neighborhood of $r = 0$, one has P-a.s. for $-\delta \le r \le \delta$,

$$M_n(\frac{r}{\sqrt{n}}) = \frac{M_n(\frac{r}{\sqrt{n}})}{\cosh^n(\frac{r}{\sqrt{n}})} \cosh^n(\frac{r}{\sqrt{n}}) \to m_\infty(0) e^{\frac{r^2}{2}} \equiv e^{\frac{r^2}{2}} \quad \text{as} \quad \mathrm{n} \to \infty.$$

Remark 9. One may in fact show with only a little more effort that the limits G and m_∞ are both a.s. analytic functions.

6 Vector Cascades

This section provides an extension of i.i.d. scalar cascades within the framework of $\mathbf{T}$-martingales. As an application an alternative approach to asymptotic path free energy calculations is given. For this, suppose that $\mathbf{W} =$

(W_1, W_2) is a symmetric random vector with a.s. positive components defined on a probability space $(\Omega, \mathcal{F}, P)$. We will denote a (scalar) random variable with the common marginal distribution of the (possibly correlated) components W_1, W_2 by W. Let

$$g(h) = E_P W^h, \quad h \in H = \{h \geq 0 : E_P W^h < \infty\}.$$

Then g is continuous on H and we restrict attention to distributions for which H is a nondegenerate subinterval of $[0, \infty)$. For $h \in H$, define

$$W_h = \frac{W^h}{g(h)} \qquad \mathbf{W}_h = (W_{h,1}, W_{h,2}) = (\frac{W_1^h}{g(h)}, \frac{W_2^h}{g(h)}).$$

For $h \in H^0$, the interior of H, one has that $\ln W$ has a finite moment generating function and, therefore,

$$E_P\left(W_h (\ln W)^n\right) < \infty, \quad \forall n = 1, 2, \ldots.$$

Moreover, from the dominated convergence theorem one has for $h \in H^0$

$$\frac{d}{dh} \ln g(h) = E_P\left(W_h \ln W\right)$$

and

$$\frac{d^2}{dh^2} \ln g(h) = E_P\left(W_h (\ln W)^2\right) - \left(E_P(W_h \ln W)\right)^2 = \mathrm{var}_h(\ln W) \geq 0,$$

where var_h denotes variance computed with respect to the size-biased probability $dQ_h = W_h dP$. In particular $\ln g(h)$ is convex on H. In fact, the function $h \to E_P(W_h \ln W_h), h \in H^0$, is increasing since

$$\frac{d}{dh} E_P(W_h \ln W_h) = h \mathrm{var}_h(\ln W) \geq 0.$$

Thus, if W is not an a.s. constant then $h \to E_P(W_h \ln W_h), h \in H^0$ is strictly increasing.

Now suppose that $\{\mathbf{W}_v = (W_{v,1}, W_{v,2}) : v \in T\}$ is an i.i.d. tree-indexed collection of random vectors defined on the probability space $(\Omega, \mathcal{F}, P)$ distributed as $\mathbf{W}$. Let

$$W_{v,(h,i)} = \frac{W_{v,i}^h}{g(h)}, \quad v \in T, i = 1, 2, h \in H,$$

and define

$$Q_n^{(h)}(t) = \prod_{j=1}^{n} W_{t|(j-1),(h,t_j)}, \quad t \in \partial T.$$

Then $\{Q_n^{(h)} : n \geq 1\}$ defines a positive $\mathbf{T}$-martingale in the sense of Kahane.

We will require a few lemmas based on the size biasing theory of section 3. Let us denote the size bias probabilities corresponding to the $\mathbf{T}$-martingale $\{Q_n^{(h)} : n \geq 1\}$ by $P_{h,t}$ and $\mathcal{Q}_h$, accordingly. The first is a law of large numbers under the size bias change of measures.

Proposition 5. *Let* $h, h' \in H$ *and* $t \in \partial T$.

1. $P_{h,t}$*-a.s.,* $\frac{1}{n} \ln Q_n^{(h')}(t) \to E_P(W_h \ln W_{h'})$. *Moreover, if there is a* $h'' \in H$ *such that* $h < h''$ *then*

$$\sum_{n=1}^{\infty} E_{P_{h,t}} \left(\frac{1}{n} \ln Q_n^{(h')}(t) - E_P(W_h \ln W_{h'}) \right)^4 < \infty.$$

2. $\mathcal{Q}_h$*-a.s.,* $\frac{1}{n} \ln Q_n^{(h')} \circ \pi_{\partial T}(t) \to E_P(W_h \ln W_{h'})$. *Moreover,*

$$\sum_{n=1}^{\infty} E_{\mathcal{Q}_h} \left(\frac{1}{n} \ln Q_n^{(h')} \circ \pi_{\partial T}(t) - E_P(W_h \ln W_{h'}) \right)^4 < \infty.$$

Proof. From the definitions one has

$$\frac{1}{n} \ln Q_n^{(h')}(t) = \frac{1}{n} \sum_{j=1}^{n} \ln W_{t|j-1,(h',t_j)}$$

is a sample average of i.i.d. terms under $P_{h,t}$ with mean $E_P(W_h \ln W_{h'})$. Thus the first assertion of of the first statement is merely a version of the strong law of large numbers and the second assertion of the first statement is the 4th moment Borel-Cantelli condition for the strong law of large numbers. Specifically, under the condition $h < h'' \in H$, one has

$$E\left(W_h (\ln W_{h'})^4\right) = E\left(W_h (h' \ln W - \ln g(h')\right)^4 < \infty.$$

For the second statement, observe that the first statement is true for λ-a.e. $t \in \partial T$. Also, by symmetry,

$$E_{\mathcal{Q}_h} \left(\frac{1}{n} \ln Q_n^{(h')} \circ \pi_{\partial T} - E(W_h \ln W_{h'}) \right)^4 = E_{P_{h,t}} \left(\frac{1}{n} \ln Q_n^{(h')}(t) - E_P(W_h \ln W_{h'}) \right)^4,$$

and is therefore also summable in n.

Lemma 5. *If* $E_P(W_h \ln W_h) < \ln 2$ *and* $h' \in H$ *then a.s.*

$$\liminf_{n \to \infty} \frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t) \lambda(dt) \geq E_P \left(W_h \ln \frac{W_{h'}}{W_h} \right).$$

Moreover, if h_c *exists such that* $E_P(W_{h_c} \ln W_{h_c}) = \ln 2$*, then a.s. one has*

$$\liminf_{n \to \infty} \frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t) \lambda(dt) \geq E_P \left(W_{h_c} \ln \frac{W_{h'}}{W_{h_c}} \right).$$

Proof. Using the size-bias change of measure one has

$$\frac{\int_{\partial T} Q_n^{(h')}(t)\lambda(dt)}{\int_{\partial T} Q^{(h)}(t)\lambda(dt)} = \frac{\int_{\partial T} \frac{Q_n^{(h')}(t)}{Q_n^{(h)}(t)} Q_n^{(h)}(t)\lambda(dt)}{\int_{\partial T} Q^{(h)}(t)\lambda(dt)}$$
$$= E_{\mathcal{Q}}\left(E_{\mathcal{Q}}(\frac{Q_n^{(h')}}{Q_n^{(h)}} \circ \pi_{\partial T} | \mathcal{F}_n)\right). \tag{9}$$

Using convexity of $x \to -\ln x$, one has

$$\frac{1}{n}\ln \int_{\partial T} Q_n^{(h')}(t)\lambda(dt)$$
$$\geq E_{\mathcal{Q}}(\frac{1}{n}\frac{Q_n^{(h')}}{Q_n^{(h)}} \circ \pi_{\partial T}|\mathcal{F}_n) + \frac{1}{n}\ln \int_{\partial T} Q_n^{(h)}(t)\lambda(dt)$$
$$= E_{\mathcal{Q}}(\frac{1}{n} Q_n^{(h')} \circ \pi_{\partial T} - \frac{1}{n}\ln Q_n^{(h)} \circ \pi_{\partial T}|\mathcal{F}_n). \tag{10}$$

Using the 4th moment summability of the second part of the previous proposition, it folllows that

$$E_{\mathcal{Q}}(\frac{1}{n} Q_n^{(h')} \circ \pi_{\partial T} - \frac{1}{n}\ln Q_n^{(h)} \circ \pi_{\partial T}|\mathcal{F}_n) \to E_P(W_h \ln W_{h'}) - E_P(W_h \ln W_h).$$

Thus, the asserted lower bound holds $\mathcal{Q}$-a.s. But, $E_P(W_h \ln W_h) < \ln 2$ implies that $\mathcal{Q} \circ \pi_{\partial T}^{-1} << P$. This proves the first assertion of the lemma. The second assertion follows from continuity of $h \to E_P(W_h \ln \frac{W_{h'}}{W_h})$.

Lemma 6. *If $E_P(W_h \ln W_h) < \ln 2$ and $h' \in H$ then a.s.*

$$\liminf_{n\to\infty} \frac{1}{n}\ln \int_{\partial T} Q_n^{(h')}(t)\lambda(dt) \leq E_P\left(W_h \ln \frac{W_{h'}}{W_h}\right).$$

Moreover, if h_c exists such that $E_P(W_{h_c} \ln W_{h_c}) = \ln 2$, then a.s. one has

$$\liminf_{n\to\infty} \frac{1}{n}\ln \int_{\partial T} Q_n^{(h)}(t)\lambda(dt) \leq E_P\left(W_{h_c} \ln \frac{W_h}{W_{h_c}}\right).$$

Proof. For any $t \in \partial T$ one has the Chebyshev bound

$$P\left(\prod_{j=1}^n W_{t|j-1,t_j} \geq c^n\right) \leq E_P\left(\frac{\prod_{j=1}^n W_{t|j-1,(h,t_j)}}{c^{hn}}\right) = (\frac{g(h)}{c^h})^n.$$

The right side is minimized at the Legendre transform value $\frac{d}{dh}(\ln g(h) - h\ln c) = 0$. In other words, $E_P(W_h \ln W) = \ln c$ optimizes to the extent that

$$P\left(\prod_{j=1}^n W_{t|j-1,t_j} \geq e^{nE_P(W_h \ln W)}\right) \leq e^{-nE_P(W_h \ln W_h)}.$$

Thus, for $h, h' \in H$, one has

$$P\left(\prod_{j=1}^{n} W_{t|j-1,(h',t_j)} \geq e^{nE_P(W_h \ln W_{h'})}\right) \leq e^{-nE_P(W_h \ln W_h)}.$$

In particular, for h_c defined by

$$E_P(W_{h_c} \ln W_{h_c}) = \ln 2,$$

one has by monotonicity that

$$E_P(W_h \ln W_h) > \ln 2 \quad \text{for} \quad h > h_c.$$

Thus

$$\sum_{n=1}^{\infty} 2^n P\left(\prod_{j=1}^{n} W_{t|j-1,(h',t_j)} \geq e^{nE_P(W_h \ln W_{h'})}\right) < \infty$$

and therefore

$$P\left(\cup_{N=1}^{\infty} \cap_{n \geq N}^{\infty} [\prod_{j=1}^{n} W_{t|j-1,(h',t_j)} < e^{nE_P(W_h \ln W_{h'})} \forall t \in \partial T\right) = 1.$$

Now consider $h' > h_c$. Then

$$\begin{aligned} &\limsup \frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t)\lambda(dt) \\ = &\limsup \frac{1}{n} \ln \int_{[Q_n^{(h')}(t) < e^{nE_P(W_h \ln W_{h'})}]} Q_n^{(h')}(t)\lambda(dt). \end{aligned}$$

Although $E_P \ln W < 0$ may not be finite, $\lim_{h \downarrow 0} E_P(W_h \ln W) = E_P(\ln W)$. Also $h \to E_P(W_h \ln W)$ is continuous and increasing on H^0. Let $0 < h_1 < h_2 < \cdots < h_{m-1} < h_c < h_m < h'$, and $c_i = \exp E_P(W_{h_i} \ln W)$. Consider the random set

$$A_{n,i} = \{t \in \partial T : c_{i-1}^n < \prod_{j=1}^{n} W_{t|j-1,t_j} \leq c_j^n\}.$$

For all n large one as a.s. that

$$\frac{c_1^m}{g^n(h')} + \sum_{i=2}^{n} \frac{c_i^m}{g^n(h')} \lambda(A_{n,i}) \geq \int_{\partial T} Q_n^{(h')}(t)\lambda(dt).$$

In any case, for $i \geq 2$ one has

$$\frac{c_{i-1}^m}{g^n(h_{i-1})} \lambda(A_{n,i}) \leq \int_{\partial T} Q_n^{(h_{i-1})}(t)\lambda(dt).$$

Thus, P-a.s.,

$$\liminf \frac{1}{n} \ln \left(\frac{c_1^m}{g^n(h')} + \sum_{i=2}^{n} \frac{c_i^{nh'} g^n(h_{i-1})}{c_{i-1}^{nh_{i-1}} g^n(h')} \int_{\partial t} Q_n^{(h_{i-1})}(t)\lambda(dt) \right)$$
$$\geq \liminf \frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t)\lambda(dt).$$

Since for $0 < h < h_c$, $\lim_n \int_{\partial T} Q_n^{(h)}(t)\lambda(dt)$ exists and is positive, one has

$$\liminf \frac{1}{n} \ln \left(\frac{c_1^m}{g^n(h')} + \sum_{i=2}^{n} \frac{c_i^{nh'} g^n(h_{i-1})}{c_{i-1}^{nh_{i-1}} g^n(h')} \int_{\partial t} Q_n^{(h_{i-1})}(t)\lambda(dt) \right)$$
$$= \ln \max \left\{ \frac{c_1^{h'}}{g(h')}, \frac{c_i^{nh'} g^n(h_{i-1})}{c_{i-1}^{nh_{i-1}} g^n(h')} \right\}$$
$$= \max \left\{ E_P(W_{h_1} \ln W_{h'}), E_P(W_{h_{i-1}} \ln W_{h'}) - E_P(W_{h_{i-1}} \ln W_{h_{i-1}}), i \geq 2 \right\}.$$

Thus, one has P-a.s. that

$$\liminf \frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t)\lambda(dt)$$
$$\leq \inf_{0<h_1<\cdots<h_{m-1}<h_c<h_m<h'} \max\{E_P(W_{h_1} \ln W_{h'}),$$
$$E_P(W_{h_{i-1}} \ln W_{h'}) - E_P(W_{h_{i-1}} \ln W_{h_{i-1}}), i \geq 2\}.$$

Now use the uniform continuity of $E_P(W_x \ln W_{h'}) - E_P(W_y \ln W_y)$ for $x, y \in [h_1, h']$ together with the fact that $E_P(\ln W_{h'}) = E_P(W_0 \ln \frac{W_{h'}}{W_0})$ to proceed as follows:

$$\inf_{0<h_1<\cdots<h_{m-1}<h_c<h_m<h'} \max\{E_P(W_{h_1} \ln W_{h'}),$$
$$E_P(W_{h_{i-1}} \ln W_{h'}) - E_P(W_{h_{i-1}} \ln W_{h_{i-1}}), i \geq 2\}$$
$$\leq \inf_{0<h_1<h_c} \max\{E_P(W_{h_1} \ln W_{h'}), \sup_{h_1<h<h_c} [E_P(W_h \ln W_{h'}) - E_P(W_h \ln W_h)\}$$
$$\leq \max\left\{ E_P(\ln W_{h'}), E_P(W_{h_c} \ln \frac{W_{h'}}{W_{h_c}}) \right\} = E_P(W_{h_c} \ln \frac{W_{h'}}{W_{h_c}}).$$

Finally, if $h' \in H$ and $h' > h_c$, then P-a.s.,

$$\liminf \frac{1}{n} \ln \int_{\partial T} Q_n^{(h')}(t)\lambda(dt) \leq E_P(W_{h_c} \ln \frac{W_{h'}}{W_{h_c}}).$$

This completes the derivation of the upper bound.

Combining these lemmas one arrives at the following result.

Theorem 2. *For $h \in H$ and $h > h_c$ one has P-a.s.*

$$\liminf_{n\to\infty} \frac{1}{n} \ln \int_{\partial T} Q_n^{(h)}(t)\lambda(dt) = -E_P\left(W_{h_c} \ln \frac{W_{h_c}}{W_h} \right).$$

Remark 10. Notice that if $h > h_c$ then it follows from the previously noted monotonicity that $E_P(W_h \ln W_h) > \ln 2$.

To apply this to the polymer model let $X_{v,j}, v \in T, j = 1, 2$, be i.i.d. positive random variables distributed as X. Assume that H is a nondegenerate interval for X defined by

$$g(h) = E_P X^h < \infty, \quad h \in H \subseteq [0, \infty).$$

Also, suppose that Y is a symmetric Bernoulli ±1-valued random variable, independent of X, and define

$$g(r,h) = E_P(e^{rY} X^h) = g(h)\cosh r, \quad h \in H, r \geq 0,$$

and consider the vector cascade weights

$$\mathbf{W}_{(r,h)} = \left(\frac{e^{rY}X_1^h}{g(r,h)}, \frac{e^{-rY}X_2^h}{g(r,h)}\right) = \left(W_{(r,h),1}, W_{(r,h),2}\right).$$

Note that defining

$$Y_r = \frac{e^{rY}}{\cosh r},$$

one has $E_P(Y_r \ln Y_r) = r\tanh r - \ln\cosh r$. Moreover, one has

$$\left\{(2\cosh r)^n \int_{\partial T} Q_n^{(r,h)}(t)\lambda(dt) : n \geq 1\right\} =^{\text{dist}} \left\{\int_{\partial T} e^{r(t)_n} Q_n^{(h)}(t)\lambda(dt) : n \geq 1\right\}.$$

With this one may obtain the following equivalent representation of the asymptotic path free energy under strong disorder.

Theorem 3. *Suppose that for $r \geq 0, h \in H, h > h_c$, there is a unique pair $(r,h)^* \equiv (r^*, h^*)$ such that $(r^*, h^*) = \gamma(r,h)$ for some $0 < \gamma < 1$, and*

$$E_P\left(W_{(r^*,h^*)} \ln W_{(r^*,h^*)}\right) \equiv E_P\left(X_{h^*} \ln X_{h^*}\right) + E_P\left(Y_{r^*} \ln Y_{r^*}\right) = \ln 2,$$

where $Y_r = \frac{e^{rY}}{\cosh r}$. Then P-a.s. one has

$$\lim_{n\to\infty} \frac{1}{n} \ln \frac{\int_{\partial T} e^{r(t)_n} Q_n^{(h)}(t)\lambda(dt)}{\int_{\partial T} Q_n^{(h)}(t)\lambda(dt)} = \ln\cosh r + E_P\left(W_{(r,h)^*} \ln W_{(r,h)}\right) - E_P\left(W_{(0,h)^*} \ln W_{(0,h)}\right).$$

Proof. The proof is essentially an application of the previous theorem using the fact that the limit has already been shown to exist. More specifically, one has P-a.s. that

$$\begin{aligned}\lim \frac{1}{n} \ln \int_{\partial T} Q_n^{(r,h)}(t)\lambda(dt) &= -E_P(W_{(r,h)^*} \ln W_{(r,h)^*} + E_P(W_{(r,h)^*} \ln W_{(r,h)}) \\ &= -\ln 2 + E_P(W_{(r,h)^*} \ln W_{(r,h)}).\end{aligned}$$

Thus

$$\lim \frac{1}{n} \ln \frac{\int_{\partial T} Q_n^{(r,h)}(t)\lambda(dt)}{\int_{\partial T} Q_n^{(0,h)}(t)\lambda(dt)} = -E_P(W_{(r,h)^*} \ln W_{(r,h)}) + E_P(W_{(0,h)^*} \ln W_{(0,h)}).$$

Now

$$\lim \frac{1}{n} \ln \frac{\int_{\partial T} e^{r(t)_n} Q_n^{(r,h)}(t)\lambda(dt)}{\int_{\partial T} Q_n^{(0,h)}(t)\lambda(dt)} = \ln\cosh r + E_P(W_{(r,h)^*} \ln W_{(r,h)} - E_P(W_{(0,h)^*} \ln W_{(0,h)})$$

as asserted.

In the strong disorder case $E_P X \ln X > \ln 2$, normalized to $E_P X = 1$, one has $h_c < 1$ by monotonicity of $h \to E_P X^h \ln X^h$. Taking $h = 1$ in this theorem gives the alternative path free energy formula. Namely,

Corollary 2. *Assume X is a positive random variable normalized to $E_P X = 1$ such that $E_P X \ln X > \ln 2$; i.e., strong disorder. Then,*

$$\lim_{n\to\infty} \frac{1}{n} \ln \int_{\partial T} e^{r(t)_n} prob_n(dt) = \ln\cosh r + E_P\left(W_{(r,1)^*} \ln W_{(r,1)}\right) - E_P\left(W_{(0,1)^*} \ln W_{(0,1)}\right).$$

7 Related directions in T-martingale theory

T-martingale theory and size bias methods occupy a central role in determining sharp results for the existence and in the analysis of the fine scale structure of diverse models; see Kahane (2000) for a review of general theory and other applications. Most of the theory, however, is devoted to analysis of fine scale structure in the weak disorder regime. Tree polymers present entirely new challenges to the theory in the case of strong disorder, and naturally motivate new directions. On the purely mathematical side, the contemplation of a companion theory for **complex T martingales** on manifolds suggests a number of new and interesting challenges for example.

In the context of tree polymer models, sharp determination of the a.s. probability laws governing polymer paths under weak and strong disorder should eventually evolve. One may not expect surprises under weak disorder but, as illustrated in the present paper for the a.s. CLT, the techniques and estimates may be delicate in the full range of weak disorder. The limits on Bolthausen's L^2 approach to a CLT for tree polymers can only be asserted when such a CLT has been established as has been achieved here. It seems to be generally accepted that the lattice polymer approach of Comets and Yoshida (2006) would also provide the CLT for tree polymers in the full range of weak disorder, but such a proof has not been available in the literature.

As is evidenced here, there is a huge amount of symmetry present both in the tree and in the environment. In general there seems to be much to understand about how and when symmetry breaking may occur. A loosely related phenomena illustrating symmetries was observed in Waymire and Williams (1995) in the consideration of a Markovian environment (along tree paths); such results were also considered by Fan (2002). In Waymire and Williams (1995), the authors demonstrate that for finite state time-reversible ergodic Markov environments, the structure of the multiplicative cascade coincides with that of i.i.d. environments distributed according to the unique invariant probability. However, examples are provided to show this is no longer true for non-reversible Markov chains.

While the emphasis of this paper is that of the theory of **T**-martingales, it is widely recognized that results obtained for branching random walks originating in Kingman (1975) and Biggins (1976) closely parallel this development. So it is not surprising that both theoretical frameworks can be applicable to tree polymers. The very recent paper by Hu and Shi (2009) illustrates many aspects of the continued development of this companion framework. In particular Hu and Shi (2009) analyze the free energy type calculations for polymers on Galton–Watson trees within the branching random walk framework. It seems natural by extension to consider the polymer path laws in the Galton–Watson environment within either framework; e.g., see Peyrière (1977) and Burd and Waymire (2000) for some multiplicative cascade theory on Galton–Watson trees.

In addition to providing a complete and self-contained diffusive limit for tree polymers in the full range of weak disorder, the goal of this paper has been to suggest new directions for extensions of the multiplicative cascade theory. The extension of Proposition 2 to corresponding P-a.s. weak limit points under strong disorder aptly illustrates such a need.

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