Transformation of Stochastic Recursions and Critical Phenomena in the Analysis of a Class of Mean Flow Equations

Radu Dascaliuc* Tuan N. Pham† Enrique Thomann‡
Edward C. Waymire§

Abstract

The purpose of this paper is to investigate, via methods generally falling under "probability on trees", critical phenomena in stochastic cascade models of Yule type, and to apply these methods to the problem of uniqueness and nonuniqueness of solutions to the linear pantograph and non-linear α-Riccati mean flow equations. For linear equations, Feller's classic discovery of the relationship between critical stochastic explosion phenomena for Markov processes and uniqueness and nonuniqueness of solutions to the associated Kolmogorov equation partly illustrates the spirit of the present paper. The connection between shocks in Burgers equation and statistical phenomena of spontaneous magnetization criticality discovered by C.M. Newman (1986) provides an illustration of another type; see [6,9,21]. New methods are introduced to mathematically explore the effects of stochastic critical phenomena, e.g., stochastic explosion, hyperexplosion, t-leaf percolation, on the mean flow. In particular, stochastic (cascade) recursions, stochastic Picard iterations and additive transforms are developed which, in certain critical parameter regimes, are shown to lead to a 'one-to-many principle' for solutions to the α-Riccati equation and a corresponding pantograph equation, related by linearization. The development also includes an application of Feller's classical theory of jump Markov processes to a certain class of pantograph equations that appears to be new.

Contents

*Department of Mathematics, Oregon State University, Corvallis, OR, 97331. dascalir@math.oregonstate.edu
†Faculty of Math and Computing, Brigham Young University - Hawaii, Laie, HI 96762. tpham@byuh.edu
‡Department of Mathematics, Oregon State University, Corvallis, OR, 97331. thomann@math.oregonstate.edu
§Department of Mathematics, Oregon State University, Corvallis, OR, 97331. waymire@math.oregonstate.edu

1 Introduction and Preliminaries

The seminal paper by Kato and McCleod [18] revealed the fascinating nature of *regularity* in the form of well-posedness (uniqueness/non-uniqueness) phenomena for the classic one-dimensional *pantograph* linear differential equation given, for parameter $\alpha > 0$, real coefficients a, b, $(a \neq 0)$ and initial value u_0, by

$$u'(t) = bu(t) + au(\alpha t), t > 0, \quad u(0) = u_0. \tag{1.1}$$

Perhaps less surprising, yet striking, revelations were shown by Athreya in [2] to occur for the *α-Riccati*[*] one-dimensional non-linear differential equation given, for parameter $\alpha \geq 0$ and initial value u_0, by

$$u'(t) = -u(t) + u^2(\alpha t), t > 0, \quad u(0) = u_0. \tag{1.2}$$

The pantograph equation (1.1) enjoys a remarkablemnumber of diverse applications in statistical physics, applied mathematics, analysis, number theory, graph theory and combinatorics, e.g., see [23] and references therein. Similarly, the α-Riccati equation (1.2) appears as a model for *data compression* in [1], *cellular senescence* in [3], and *fluid flow* in [8, 11, 12, 14].

The interest in these equations by the present authors is mostly based on a mean-field heuristic (see [14], p.55) in which the α-Riccati equation results from self-similar/rotational symmetry and scaling considerations of the three-dimensional incompressible Navier-Stokes equations. The essential idea is derived from the random cascade discovered by [19] for the mild form of the Fourier transformed equations. Namely, in allowing for possible stochastic explosion by suppression of

[*]The term *α-Riccati* was introduced by the authors in deference to the standard Riccati equation ($\alpha = 1$). The corresponding stochastic model is sometimes also referred to as the *Aldous-Shields model* [3, 15], *generalized Eden growth* [16], *discounted branching random walk* [2], or an *inhomogenous Yule model* as here. The differential equation appears to have otherwise been nameless in the existing literature.

their coin-toss mechanism, the α-Riccati equation is obtained for their *self-similar* probability kernel $\frac{1}{\pi^3|\xi|^2}, \xi \in \mathbb{R}^3\backslash\{0\}$, by replacing the random multiplicative factors by the parameter α, and replacing the bilinear vector product at branching by ordinary multiplication.

Likewise, the pantograph equation (1.1), with $b < 0$ can be viewed as a linear counterpart of the α-Riccati equation. Indeed, via a change or variables, (1.1) can be written in the form

$$u'(s) = -u(t) + au(\alpha t), t > 0, \quad u(0) = u_0, \tag{1.3}$$

which admits a stochastic structure similar to that of (1.2), except the product at branching is replaced by a weighted sum. Alternatively, the linear nature of (1.3) allows a stochastic cascade approach based on a unary tree structure, placing it within a more classical framework of jump Markov processes and associated Kolmogorov backwards equations, as shown in Section 2.

In Section 4 we will investigate several critical phenomena related to the above-mentioned stochastic structures viewed via a general framework of the Doubly Stochastic Yule (DSY) processes introduced in [11, 12]. We particularly focus on the size of the t-*leaf* sets and on existence of so called *hyper-exploding subtrees* of DSY trees – properties that will be crucial in establishing the richness of non-uniqueness of the solutions for both (1.3) and (1.2). In particular, the critical nature of the case $\alpha = 2$ is revealed within the explosion range $\alpha \in (1, \infty)$.

For the sake of clarity, unless otherwise stated, by solutions, we mean *global solutions* to the initial value problems (1.2) or (1.3), i.e. solutions that exist for all $t > 0$ and satisfy the initial condition as $t \to 0$. Note that such solutions are necessarily C^∞ on $t > 0$ (in fact, if $\alpha \in (0, 1]$, the solutions are analytic).

In Section 5 we describe the key approach – the *stochastic Picard iterations* used to build solution processes whose expectations solve the integral form of (1.3) and (1.2). This approach is inspired by a martingale technique of [19] for uniqueness of solutions to the Navier-Stokes equations in appropriate functional setting, and was first introduced in [14] and subsequently exploited in [10] to prove several existence and non-uniqueness results for Cauchy problems for pde's related to the $3d$-incompressible Navier-Stokes equations.

Note that in the the case $a = 2$, (1.3) is a linearization of (1.2) around the steady state $u = 1$, In Section 6 we will establish a connection between (1.2) and (1.3) with $a = 2$ *at the level of stochastic structures* that allows not only to deduce the existence of non-unique solutions of (1.2) for a range of initial data, but also to determine their long-time behavior.

Finally, in Section 6.1 we present several numerical simulations illustrating the results of Section 6.

Equations (1.3) and (1.2) primarily serve as mathematical surrogates[†] for aspects of the regularity theory of differential equations amenable to probabilistic methods of analysis. In particular, certain critical phenomena associated with the stochastic model are shown to have significant consequences for the regularity of these equations.

While the focus of this paper is on the equations (1.3) and (1.2), the broader purpose is to illustrate an emerging theory for classes of non-linear partial differential equations of the type represented by the incompressible Navier-Stokes equations based on contemporary methods from

[†]Here "surrogate model" is intended to be in the spirit of the idealized Ising model in statistical physics, logistic model in population biology, discrete Gaussian free field in quantum field theory, etc.

“probability on trees”, e.g. see [20]. In particular, such notions of *stochastic explosion*, *hyper-explosion*, *t-leaf percolation* criticalities, and methodologies of *stochastic Picard iteration*[‡] *from ground states*, *stochastic transformations* and a *one-to-many solution principle* are introduced and applied to the pantograph and α-Riccati equations.

The main results begin with a new,[§] albeit constrained approach to (1.3) based on self-similarity and Feller's classical theory of Markov processes and semi-groups in the next section. This also motivates a transition to a less constrained approach via probability on trees. From here the remainder of the paper is devoted to the development of special techniques to analyze the well-posedness of these equations. In comparison with Kato-McLeod [18], we show existence of a family of solutions to pantograph (for any given initial data) with $\alpha > 1$, $a > -b > 0$ that have an algebric as $t \to \infty$, see Theorem 3.5.

2 Self-Similarity and Probability on Trees

The main idea behind stochastic representations of solutions of equations such as (1.3) and (1.2), is that their mild-type formulation can be connected to the expected values of certain *progressively measurable* stochastic processes – *solution processes* defined on a suitable probability space.

The "self-similar" nature of both pantograph and α-Riccati equations is revealed by consideration of the following evolution PDEs:

$$\frac{\partial v}{\partial t}(\ell,t) = -\ell v(\ell,t) + a\ell v(\alpha\ell,t), \quad \ell > 0, t \geq 0, \quad v(\ell,0) = v_0(\ell) \tag{2.1}$$

and

$$\frac{\partial v}{\partial t}(\ell,t) = -\ell v(\ell,t) + \ell v^2(\alpha\ell,t), \quad \ell > 0, t \geq 0, \quad v(\ell,0) = v_0(\ell). \tag{2.2}$$

We refer to these equations as the *space-time* counterparts of (1.3) and (1.2), respectively. The use of the parameter $\ell \in (0,\infty)$ is intentional, as it is merely a mathematical *label* without special physical significance otherwise. However, one can view (2.1) and (2.2) as a non-local differential equation in Fourier space with $\ell = |\xi|$.

The system (2.1) has a natural symmetry: if $v(t,\ell)$ is a solution, then $v_\lambda(t,\ell) = v(\ell/\lambda, \lambda t)$ is also a solution. This permits one to consider solutions of the *self-similar* form $v(\ell,t) = u(\ell t)$, where the product $s = \ell t$ defines a *similarity* variable, and u solves (1.3) and (1.2), respectively.

It is often convenient to express the space-time equations (2.1) and (2.2) in an integral form as follows:

$$v(\ell,t) = e^{-\ell t}v_0(\ell) + a\int_0^t \ell e^{-\ell s} v(\alpha\ell, t-s)ds, \quad \ell > 0, t \geq 0. \tag{2.3}$$

and

$$v(\ell,t) = e^{-\ell t}v_0(\ell) + \int_0^t \ell e^{-\ell s} v^2(\alpha\ell, t-s)ds, \quad \ell > 0, t \geq 0. \tag{2.4}$$

[‡]A familiar, though mostly unrelated, application of stochastic Picard iteration is in proofs of existence and uniqueness of solutions to stochastic differential equations with Lipschitz coefficients.

[§]As far as the authors can determine, this approach to well-posedness via self-similarity and Feller's jump Markov process theory appears to be new for the pantograph equations.

Note that self-similar forms of (2.3) and (2.4) are the corresponding mild-type formulations of (1.3) and (1.2):

$$u(t) = e^{-t}u_0 + a\int_0^t e^{-s}u(\alpha(t-s))ds, \quad \ell > 0, t \geq 0. \tag{2.5}$$

and

$$u(t) = e^{-t}u_0 + \int_0^t e^{-\ell s}u^2(\alpha(t-s))ds, \quad \ell > 0, t \geq 0, \tag{2.6}$$

where $u(s) = v(\ell,t)|_{\ell=1,t=s} = v(\ell,t)|_{\ell=s,t=1}$, $u_0 = v_0(1)$, for a self-similar solution v to (1.3) and (1.2), respectively.

2.1 A Probabilistic Framework for the Pantograph Equation

The self-similar embedding of the classic pantograph equation into a space-time pantorgraph equation yields a probabilistic framework in which one may view (2.1) as a Kolmogorov backward equation for a jump Markov process. To keep the comparisons simple, let us assume $a = 1$, and $v_0 = 1$ in (2.3) and (2.5) for now. Then, in this case, the corresponding Markov process holds in state $\ell \geq 0$ for an exponential time of intensity $\lambda(\ell) = \ell$, 0 being absorbing, before transitioning to state $\alpha\ell$, i.e., with transition probability kernel $k(\ell_1, d\ell_2) = \delta_{\alpha\ell_1}(d\ell_2)$. The constant solution $v(\ell,t) \equiv 1$ for all $\ell, t \geq 0$ is obviously a solution (for $v_0 = 1$), and is unique if and only if there is no explosion. Now, the jump Markov process starting at $\ell > 0$ is clearly explosive if and only if $\alpha > 1$ since the mean time between the n-th and $(n+1)$-st jumps is $\frac{1}{\ell\alpha^n}$ for $n \geq 0$. In fact, in this case for any initial state ℓ the *explosion time* random variable $\tilde{S}_\ell = \sum_{j=0}^{\infty}\frac{T_j}{\ell\alpha^j} < \infty$ a.s. (here $\{T_j\}$ are iid exponential random variables with intensity 1, $\frac{T_j}{\ell\alpha^j}$ representing the time between jumps j and $j+1$). Accordingly, for $\alpha > 1$ the minimal jump process may be continued beyond explosion time by absorbing the process in an adjoined spatial point at infinity. From here it is standard Markov process theory, e.g., see [4, 5], that this transformation does not alter the infinitesimal behavior of (2.1) with $a = 1$. In particular, the equation is satisfied by the substochastic transition probabilities $p(1; \ell_1, d\ell_2)$ of the minimal process and, hence,

$$u(s) = v(\ell,t)\big|_{\ell=s,t=1} = \int_{[0,\infty)} v(\ell_2, 0)p(1; s, d\ell_2) = p(1; s, [0,\infty)) < 1$$

is a so-called *minimal solution* to (2.5) (and, consequently, for (1.3)) with $a = 1$, $u_0 = 1$, which in the case $\alpha > 1$ is distinct for the steady state $u \equiv 1$. Note also that this minimal solution has an explicit representation in terms the aforementioned explosion time $\tilde{S}_{\ell=s} = \sum_{j=0}^{\infty}\frac{1}{s\alpha^j}T_j$:

$$u(s) \equiv \mathbb{P}_{\ell=s}(\tilde{S}_s > 1) = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\frac{e^{-\alpha^j s}\alpha^{j-n}}{\prod_{\substack{k=\overline{0,n}\\ k\neq j}}(1-\alpha^{j-k})}. \tag{2.7}$$

The series expansion is the complementary cdf for an infinite sum of independent exponentially distributed random variables with distinct intensities $\alpha^j s, j = 0, 1, \ldots$ for $\alpha > 1$; see ([17], p.40, #12) for convolution formula for non-identical exponential distributions. Using this, (2.7) follows from the following calculation of the distribution of the number $\tilde{N}_\ell(t)$ of clock rings by time t with initial state ℓ by setting $t = 1, \ell = s$:

$$\begin{aligned}\mathbb{P}\left(\tilde{N}_\ell(t) = n\right) &= \mathbb{P}\left(\sum_{j=0}^{n} \frac{1}{\ell\alpha^j} T_j > t\right) - \mathbb{P}\left(\sum_{j=0}^{n-1} \frac{1}{\ell\alpha^j} T_j > t\right) \\ &= \sum_{j=0}^{n} \frac{e^{-\alpha^j \ell t}}{\prod_{\substack{k=\overline{0,n}\\ k\neq j}}(1-\alpha^{j-k})} - \sum_{j=0}^{n-1} \frac{e^{-\alpha^j \ell t}}{\prod_{\substack{k=\overline{0,n-1}\\ k\neq j}}(1-\alpha^{j-k})} \\ &= \sum_{j=0}^{n} \frac{e^{-\alpha^j \ell t}\alpha^{j-n}}{\prod_{\substack{k=\overline{0,n}\\ k\neq j}}(1-\alpha^{j-k})}. \end{aligned} \tag{2.8}$$

The formula (2.7) is equivalent to the series expansion obtained in the aforementioned paper [18] by analytic methods in the cases that one may exchange the order of summation there.

Remark 2.1. In the non-explosive case $\alpha < 1$, it follows from uniqueness of solutions to the backward equation that the sum over all n in (2.8) is identically one for all $t \geq 0$. This is obvious in the case $\alpha = 1$ since the convolution of identical exponential distributions is a Gamma distribution and the formula (2.8) is a Poisson distribution. However the sum over n is strictly less than one for all $t, \ell > 0$ if $\alpha > 1$.

More generally, if the coefficient $0 < a \leq 1$ in (1.3) and (2.3), then consider the jump Markov process on the compactified half-line $[0, \infty]$ with $\lambda(\ell) = \ell, \ell < \infty$, $\lambda(\infty) = 0$, $k(\ell_1, d\ell_2) = a\delta_{\{\alpha\ell_1\}}(d\ell_2) + (1-a)\delta_{\{\infty\}}(d\ell_2)$ on $[0,\infty)$ and $k(\infty, d\ell_2) = \delta_{\{\infty\}}(d\ell_2)$, i.e., ∞ is absorbing. By the same arguments as above with initial data $v(\ell, 0) = 1, \ell \in (0,\infty), v(\infty, 0) = 0$, one obtains the pantograph equation for self-similar solutions $v(\ell, t)$ vanishing at $\ell = \infty$. (Note that the constant $u \equiv 1$ is no longer a solution to the pantograph equation for $a \neq 1$.) For $a = 1$ this coincides with the previous treatment, however if $0 < a < 1$ then almost surely there can only be finitely many finite jumps before a transition to infinity occurs, regardless of the value of $\alpha > 0$. That is, stochastic explosion has probability zero and the unique solution to (1.3) with $u_0 = 1$ is given by

$$u(s) = p(1; s, [0,\infty)) = \sum_{n=0}^{\infty} a^n \sum_{j=0}^{n} \frac{e^{-\alpha^j s}\alpha^{j-n}}{\prod_{\substack{k=\overline{0,n}\\ k\neq j}}(1-\alpha^{j-k})} \tag{2.9}$$

Also, since $\lambda(0) = 0$ implies that 0 is an absorbing state, in the case that $u_0 = 0$ the solution so obtained is the identically zero solution.

The above assumptions on the coefficients a, b in (1.1) are clearly an obstruction to an analysis of the full problem. However, in view of the Hille-Yosida theorem, some restriction is intrinsic to Feller's semigroup theory underlying the approach.

A closer look at the underlying stochastic structure, however, reveals a naturally occurring rooted *unary tree* with label $\ell = s > 0$, consisting of above-mentioned i.i.d. mean one exponentially distributed random variables $\{T_j\}$ scaled by $\frac{1}{\alpha^j \ell}$ to have intensities $\alpha^j \ell$. In particular, one may take $\ell = 1$ to obtain the self-similarity parameter is $\ell t = t$. See Figure 1 for a realization of this unary tree. This provides a gateway to an approach in which the tree is used to define a *stochastic recursion* underlying (1.3) with the constraint $0 < a \leq 1$ removed. Namely, let us

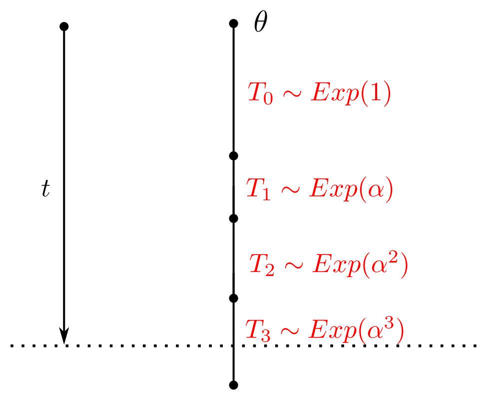

Figure 1: Pantograph unary tree

define a *(unary) stochastic solution solution process* for (1.3) as a stochastic process $\tilde{\mathfrak{X}}$ satisfying the recursion

$$\tilde{\mathfrak{X}}(t) = \begin{cases} u_0 & \text{if } T_0 \geq t \\ a\tilde{\mathfrak{X}}(\alpha(t - T_0)) & \text{if } T_0 < t. \end{cases} \tag{2.10}$$

By conditioning on T_0, it is simple to check that $u(t) = \mathbb{E}(\tilde{\mathfrak{X}}(t)\mathbb{1}_{[\tilde{S}>t]})$ satisfies (2.5), and thus provides a self-similar solution to (2.1), when the expectation exists. Here, as before, $\tilde{S} = \tilde{S}_{\ell=1}$ is the explosion time (from the initial state $\ell = 1$):

$$\tilde{S} = \sum_{j=0}^{\infty} \frac{T_j}{\alpha^j} \tag{2.11}$$

Iterating this recursion the non-explosion event $[\tilde{S} > t]$, one obtains

$$u(t) = \mathbb{E}(u_0\, a^{\tilde{N}(t)}\mathbb{1}_{[\tilde{S}>t]}) = u_0 \sum_{n=0}^{\infty} a^n \sum_{j=0}^{n} \frac{e^{-\alpha^j t}\alpha^{j-n}}{\prod_{\substack{k=\overline{0,n} \\ k\neq j}}(1-\alpha^{j-k})}, \quad t > 0, \tag{2.12}$$

Where $\tilde{N}(t) = \tilde{N}_{\ell=1}-$ is the number of clock rings before time t in the unary tree. Under the assumption that $\alpha > \max\{|a|, 1\}$, this double series can be rearranged into

$$u(t) = u_0\, c_{a,\alpha} \sum_{n=0}^{\infty} \frac{a^n e^{-\alpha^n t}}{\prod_{j=1}^{n}(1-\alpha^j)} \tag{2.13}$$

where

$$c_{a,\alpha} = \sum_{n=0}^{\infty} \frac{a^n}{\alpha^n \prod_{j=1}^{n}(1-\alpha^{-j})}$$

In this formula, the standard convention that $\prod_{j=1}^{0} = 1$, is used. The formula (2.13) is also obtained in [18] by analytic methods. Note in particular that the complementary distribution function $G(t) = \mathbb{P}(\tilde{S} > t)$ is given by

$$G(t) = C_\alpha \sum_{n=0}^{\infty} \frac{e^{-\alpha^n t}}{\prod_{j=1}^{n}(1-\alpha^j)}, \quad C_\alpha = \sum_{n=0}^{\infty} \frac{1}{\alpha^n \prod_{j=1}^{n}(1-\alpha^{-j})} \tag{2.14}$$

Remark 2.2. For $0 < a < 1$ the tree probability recursion may be modified as

$$\tilde{\mathfrak{X}}(t) = \begin{cases} u_0 & \text{if } T_0 > t \\ 0 & \text{if } C = 0, T_0 < t \\ \tilde{\mathfrak{X}}(\alpha(t - T_0)) & \text{if } C = 1, T_0 < t, \end{cases} \tag{2.15}$$

where $C \in \{0, 1\}$ is a Bernoulli fair coin tossing random variable independent of the holding time T with $\mathbb{P}(C = 1) = a$. In this iteration a^n explicitly represents the probability of n clock rings prior to absorption at 0. In particular, note that the stochastic recursion does not require compactification of the half-line.

2.2 Probabilistic Framework for α-Riccati Equations.

In the case of the α-Riccati equation in mild form (2.6), a *stochastic solution process* may be similarly defined by a recursion on the *binary* tree by

$$X(t) = \begin{cases} u_0 & \text{if } T_\theta \geq t \\ X^{(1)}(\alpha(t - T_\theta)\, X^{(2)}(\alpha(t - T_\theta)) & \text{if } T_\theta < t, \end{cases} \tag{2.16}$$

where T_θ is the root clock distributed exponentially with intensity 1, and $X^{(1)}, X^{(2)}$ are (conditionally on T_θ) independent copies of X re-rooted at $(1), (2)$, respectively.

One may note that the *(binary) stochastic solution process* for the pantograph equation (2.5) may also be defined on a binary tree via

$$\mathfrak{X}(t) = \begin{cases} u_0 & \text{if } T_\theta \geq t \\ \frac{a}{2}\mathfrak{X}^{(1)}(\alpha(t - T_\theta)) + \frac{a}{2}\mathfrak{X}^{(2)}(\alpha(t - T_\theta)) & \text{if } T_\theta < t, \end{cases} \tag{2.17}$$

where $\mathfrak{X}^{(1)}, \mathfrak{X}^{(2)}$ are conditionally on T_θ independent copies of $\mathfrak{X}$ re-rooted at $(1), (2)$, respectively.

Let us introduce a bit of binary tree notation. Let $\mathbb{T} = \{\theta\} \bigcup (\cup_{n=0}^{\infty}\{1, 2\}^n)$ be a binary tree rooted at θ. For a vertex $v = (v_1, \ldots, v_n) \in \mathbb{T}$, let $|v| = n, v|j = (v_1, \ldots, v_j), j = 1, \ldots, n$, $v|0 = \theta$. Also, denote by $\overleftarrow{v} = v|(|v| - 1)$ – the parent of a vertex $v \in \mathbb{T} \setminus \{\theta\}$.

The essential ingredients underlying the definition of the stochastic solution processes is a tree-indexed family $\mathbf{Y} = \{Y_v := \alpha^{-|v|}T_v : v \in \mathbb{T}\}$, where $T_v, v \in \mathbb{T}$ are i.i.d. mean one exponentially distributed random variables defined on a probability space $(\Omega, \mathscr{F}, P)$, and a multiplicative scaling parameter $\alpha > 0$. See Figure 2 for a realization of this binary tree. We refer to $\mathbf{Y}$ as *inhomogeneous Yule random field* based on its implicit role in the classic Yule counting process when $\alpha = 1$. In view of the representation of solutions as expected values of the solution process (2.16) and (2.17), the equations (2.5) and (2.6) are, respectively, referred to as the *mean flow equations*. From a perspective of self-similarity, in which t is viewed as the self-similarity parameter, the rescaling by α at each exponential clock ring correspond to spatial transitions of labels $\ell \to \alpha\ell$ in (2.3) and (2.4).

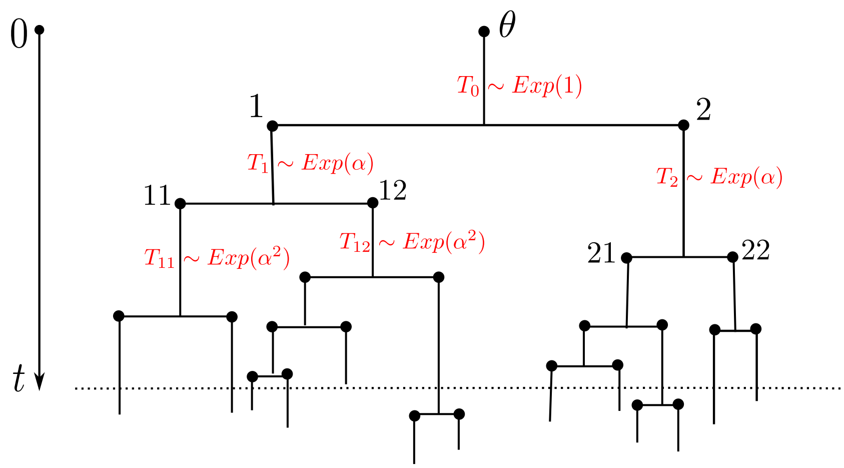

Figure 2: Binary tree corresponding to the α-Riccati Equation

3 Non-uniqueness of Solutions to the Pantograph Equation via Unary Solution Processes

We start with the following result about existence and uniqueness of solutions for (1.3) that follow directly from discussions in Section 2.1.

Proposition 3.1. *When $\alpha > \max\{1, |a|\}$, for any initial data $u_0 \in \mathbb{R}$ the minimal solution $u(t)$ to* (1.3) *defined by* (2.12) *satisfies*

$$\lim_{n\to\infty} \frac{|u(t)|}{e^{-t}} = u_0 c_{a,\alpha}. \tag{3.1}$$

Moreover, if $-1 < a \leq 1$, then for any initial data u_0, the initial value problem for pantograph equation (1.3) *has:*

(i) a unique globally bounded solution if $a = 1$ and $\alpha \in (0, 1]$ or if $|a| \in (0, 1)$ and $\alpha > 0$.

(ii) infinitely many globally bounded solutions when $\alpha > 1$, $a = 1$.

Proof. In case $\alpha > 1$, the minimal solution is given by (2.13), which yields (3.1).

Note that in the case $|a| \leq 1$, the minimal solution (2.12) is globally bounded.

In the case $a = 1$, (1.3) coincides with (2.7), as we noted earlier, $u(t) \to 0$ as $t \to \infty$ is a distinct solution corresponding to $u_0 = 1$, in addition to the steady state 1. By linearity, this already implies non-uniqueness of bounded solutions for any $u_0 \neq 0$. In the case $u_0 = 0$, consider $w(t) = 1 - u(t)$ ($u(t)$ is still given by (2.7)). Note that w is a bounded solution to (1.3) corresponding to $u_0 = 0$, distinct from the steady state 0. Thus, for any $\lambda \in \mathbb{R}$, $\lambda w(t)$ is also a bounded solution corresponding to $u_0 = 0$. This result transfers to arbitrary u_0 via linearity.

To prove uniqueness in the non-explosive case ($0 < \alpha < 1$), let $v(t)$ be a solution to (1.3), $v(0) = u_0$, $|v(t)| \leq M$ for all $t \geq 0$. Let $\tilde{\mathfrak{X}}_0(t) = v(t)$. For all $n \in \mathbb{N}$ consider the iterative scheme

$$\tilde{\mathfrak{X}}_n(t) = v\left(\alpha^n(t - \Theta_{n-1}(t))\right) a^n \mathbb{1}_{[\Theta_n < t]} + u_0 a^{\tilde{N}(t)} \mathbb{1}_{[\Theta_{n-1} \geq t]},$$

where $\Theta_n = \sum_{j=0}^n T_j/\alpha^j$. Note that $\lim_{n\to\infty}\Theta_n = \tilde{S}$ a.s.. If $\alpha \le 1$, $\tilde{S} = \infty$ a.s., so $\tilde{\mathfrak{X}}_n(t) \to u_0 a^{\tilde{N}(t)}$ a.s.. Likewise, if $a < 1$, on the event $[\tilde{S} > t]$, $\tilde{\mathfrak{X}}_n(t) \to \tilde{\mathfrak{X}}(t) = u_0 a^{\tilde{N}(t)}$. Clearly, for each n, $\mathbb{E}(|\tilde{\mathfrak{X}}_n(t)|) \le M + u_0$. In addition, $\tilde{\mathfrak{X}}(t) = u_0 a^{\tilde{N}(t)}$ satisfies (2.10), and therefore $u(t) = \mathbb{E}(\tilde{\mathfrak{X}}(t))$ is a well-defined solution of (1.3), and by the dominated convergence theorem, $v_n(t) = \mathbb{E}(\tilde{\mathfrak{X}}_n(t)) \to u(t)$ as $n \to \infty$. Moreover, note that for all every $n \ge 1$,

$$\tilde{\mathfrak{X}}_n(t) = \begin{cases} u_0 & \text{if } T_0 \ge t \\ a\tilde{\mathfrak{X}}_{n-1}(\alpha(t-T_0)) & \text{if } T_0 < t, \end{cases} \tag{3.2}$$

and therefore $v_n(t) = \mathbb{E}(\tilde{\mathfrak{X}}_n(t))$ satisfies

$$v_n(t) = u_0 e^{-t} + \int_0^t e^{-s} v_{n-1}(\alpha(t-s))\, ds$$

Note that since $v(t)$ solves (1.3), and $v_0 = v$,

$$v_1(t) = u_0 e^{-t} + \int_0^t e^{-s} v(\alpha(t-s))\, ds = v(t)\,.$$

Then, by induction, $v_n(t) = v(t)$ for all n. Hence $v(t) = u(t)$, and we have uniqueness in the class of bounded solutions.

□

Remark 3.2. In fact, in the case $a = 1$, $\alpha > 1$ Feller's theory allows for a rich variety of non-unique solutions for the same initial data simply by *instantaneously* reinitiating the underlying Markov process at the successive times of explosion at a designated state $\ell \in \mathbb{R}$.

Remark 3.3. In fact, as shown in [18], globally bounded solutions exist for all $\alpha > 0$, $\alpha, u_0 \in \mathbb{R}$.

We can "bootstrap" the previous result by integrating/differentiating (1.3).

Proposition 3.4. *A solution $u(t)$ to (1.3), with $u_0 = 0$, is infinitely differentiable, and for each integer n,*

$$w_n(t) = \begin{cases} u(t), & \text{if } n = 0; \\ \frac{d^n}{dt^n} u(t), & \text{if } n > 0; \\ \int_0^t \int_0^{t_1} \cdots \int_0^{t_{|n|-1}} u(s) ds dt_{|n|-1} \cdots dt_1, & \text{if } n < 0, \end{cases}$$

satisfy

$$\frac{d}{dt} w_n(t) = -w_n(t) + a\alpha^n w^n(\alpha t), \quad u^{(n)}(0) = 0. \tag{3.3}$$

Moreover, for $a > 0$ and $\alpha = a^{-\frac{1}{n}} > 1$ for some integer $n \le -1$, (1.3) has infinitely many solutions.

Proof. The smoothness of $u(t)$ follows e.g. from the integral representations (1.3). For $n \ge 1$, iterated differentiation yields (3.3), and for $n \le -1$, iterated integration yields the same. By Proposition 3.1, in the case $\alpha > 1$, $a = 1$, there exist infinitely many solutions of (1.3). In the case $\alpha = a^{-\frac{1}{n}} > 1$, (3.3) transfers this non-uniques result to (6.1). □

Thus, our next goal is to construct a solution process $\tilde{\mathfrak{X}}_*(t) \geq 0$ of (2.10) that is not identically zero. The relatively slow decay at infinity of $\mathbb{E}(\tilde{\mathfrak{X}}_*(t))$ is exploited in an essential way to prove the non-uniqueness result for (1.3) in Section 6.2.

Theorem 3.5. *Let $\alpha > \max\{|a|, 1\}$, $a \neq 0$ and*

$$\gamma = \gamma(a, \alpha) = \log_{|a|} \alpha = \frac{\ln |a|}{\ln \alpha} \in (0, 1). \tag{3.4}$$

Let

$$\tilde{\mathfrak{X}}_*(t) = \tilde{\mathfrak{X}}_*(t; a, \alpha) = (t - \tilde{S})^{-\gamma} \mathbb{1}_{[\tilde{S} < t]}, \tag{3.5}$$

where $\tilde{S}$ is the unary explosion time given by (2.11). Then

(i) $\tilde{\mathfrak{X}}_*$ *a.s. satisfies (2.10) with $u_0 = 0$.*

(ii) $\eta(t) = \mathbb{E}(\tilde{\mathfrak{X}}_*(t))$ *satisfies (1.3) with $u_0 = 0$.*

(iii) $\lim_{t\to\infty} \frac{\eta(t)}{t^{-\gamma}} = 1$.

Proof. Let $\tilde{S}^{(1)} = \sum_{j=1}^{\infty} \frac{T_j}{\alpha^{j-1}}$ and $\tilde{\mathfrak{X}}_*^{(1)}(\tau) = (\tau - \tilde{S}^{(1)}) \mathbb{1}_{[\tilde{S}^{(1)} < \tau]}$. Note that $\tilde{S}$ and $\tilde{S}^{(1)}$ as well as $\tilde{\mathfrak{X}}$ and $\tilde{\mathfrak{X}}_*^{(1)}$ are identically distributed. We have

$$\begin{aligned}
\tilde{\mathfrak{X}}_*(t) = (t - \tilde{S})^{-\gamma} \mathbb{1}_{[\tilde{S} < t]} &= \left(t - T_0 - \frac{1}{\alpha}\tilde{S}^{(1)}\right)^{-\gamma} \mathbb{1}_{[T_0 + \frac{1}{\alpha}\tilde{S}^{(1)} < t]} \\
&= \frac{1}{\alpha^{-\gamma}} \left(\alpha(t - T_0) - \tilde{S}^{(1)}\right)^{-\gamma} \mathbb{1}_{[\tilde{S}^{(1)} < \alpha(t - T_0)]} = a \tilde{\mathfrak{X}}_*^{(1)}(\alpha(t - T_0)) \, .
\end{aligned}$$

Note that if $T_0 \geq t$, the calculations above yield $\tilde{\mathfrak{X}}_*(t) = 0$. Thus, $\tilde{\mathfrak{X}}_*(t)$ in distribution satisfies (2.10) with $u_0 = 0$ and $a = 2$.

Using (2.14) one obtains that the *pdf* of the unary explosion time $\tilde{S}$ is given by

$$g(t) = -G'(t) = C_\alpha \sum_{n=0}^{\infty} \frac{\alpha^n e^{-\alpha^n t}}{\prod_{j=1}^{n} (1 - \alpha^{-j})}$$

since differentiation can easily be justified. As a consequence, $g(t) \sim C_\alpha e^{-t}$ as $t \to \infty$, $g(t) \leq Ce^{-t}, \forall t \geq 0$, and thus $\eta(t)$ is well defined. The fact that $\eta(t)$ satisfies (2.5) with $u_0 = 0$, follows from (2.10). More directly, note

$$\begin{aligned}
\eta(t) = \mathbb{E}(\tilde{\mathfrak{X}}_*(t)) &= \mathbb{E}\left(\left(t - T_0 - \frac{1}{\alpha}\tilde{S}^{(1)}\right)^{-\gamma} \mathbb{1}_{[\tilde{S}^{(1)} < \alpha(t - T_0)]}\right) \\
&= \alpha^\gamma \int_0^t e^{-s} \mathbb{E}\left(\left(\alpha(t - s) - \tilde{S}^{(1)}\right)^{-\gamma} \mathbb{1}_{[\tilde{S}^{(1)} < \alpha(t - s)]}\right) ds \\
&= 2 \int_0^t e^{-s} \eta(\alpha(t - s)) ds,
\end{aligned}$$

so η satisfies (1.3).

To establish *(iii)*, let $\phi(t) = t^{-\gamma}\mathbb{1}_{t>0}$ and extend g by 0 on the interval $(-\infty, 0)$. Then

$$\eta(t) = \int_0^t (t-s)^{-\gamma} g(s)ds = \int_{-\infty}^{\infty} \phi(t-s)g(s)ds$$

and

$$\frac{\eta(t)}{t^{-\gamma}} = \int_{-\infty}^{\infty} \frac{\phi(t-s)}{\phi(t)} g(s)ds = \mathbb{E}(Z(t))$$

where $Z(t) = \frac{\phi(t-\tilde{S})}{\phi(t)}\mathbb{1}_{[\tilde{S}<t]}$. Note that due to explosion, $\tilde{S} < \infty$ a.s. and so $\lim_{t\to\infty} Z(t) = 1$ a.s. We will use uniform integrability to prove that $\mathbb{E}(Z(t)) \to \mathbb{E}(1) = 1$ by showing that there exists $p \in (1,\infty)$ and $\tilde{C} > 0$ such that

$$\mathbb{E}(Z(t)^p) \le \tilde{C} \quad \forall\, t \ge 1.$$

Fix $p \in (1, \frac{1}{\gamma})$. It suffices to show that $\limsup_{t\to\infty} \mathbb{E}(Z(t)^p) < \infty$. We have

$$\begin{aligned}
\mathbb{E}(Z(t)^p) &= \mathbb{E}\left(\frac{\phi^p(t-S)}{\phi^p(t)} 1_{S<t}\right) = \int_0^t \frac{(t-s)^{-\gamma p}}{t^{-\gamma p}} g(s)ds \\
&= \int_0^t \frac{s^{-\gamma p}}{t^{-\gamma p}} g(t-s)ds \le C \int_0^t \frac{s^{-\gamma p}}{t^{-\gamma p}} e^{-(t-s)}ds \\
&= C \frac{\int_0^t s^{-\gamma p} e^s ds}{t^{-\gamma p} e^t}.
\end{aligned}$$

By L'Hospital's Rule,

$$\begin{aligned}
\limsup_{t\to\infty} \mathbb{E}[Z(t)^p] &\le C \lim_{t\to\infty} \frac{\int_0^t s^{-\gamma p} e^s ds}{t^{-\gamma p} e^t} = C \lim_{t\to\infty} \frac{t^{-\gamma p} e^t}{t^{-\gamma p} e^t - \gamma p t^{-\gamma p - 1} e^t} \\
&= C \lim_{t\to\infty} \frac{1}{1 - \frac{\gamma p}{t}} = C.
\end{aligned}$$

The proof is complete.

□

Remark 3.6. An alternative proof of (ii) in Theorem 3.5 can be obtained using Karamata Tauberian theorems since the Laplace transform of η can explicitly be computed. However, the argument using uniform integrability presented in the above proof is more direct.

Remark 3.7. Fix $a > 0$ and let $\alpha > \max\{a, 1\}$. Consider the solution process $\tilde{\mathfrak{X}}_*(t) = \tilde{\mathfrak{X}}_*(t; a, \alpha)$ from Theorem 3.5 and $\gamma = \gamma(a, \alpha)$ as in (3.4). Then, for any $\delta \in (0, 1/\gamma)$, we have $a^\delta < \alpha$, $\gamma(a^\delta, \alpha) = \delta\gamma$, and $\tilde{\mathfrak{X}}_*(t; a, \alpha)^\delta = \tilde{\mathfrak{X}}_*(t; a^\delta, \alpha)$, and thus

$$\eta_\delta(t) = \mathbb{E}(\tilde{\mathfrak{X}}_*(t, \gamma)^\delta)$$

satisfies (1.3) with $u_0 = 0$ and a replaced by a^δ:

$$\eta_\delta' = -\eta_\delta + a^\delta \eta_\delta(\alpha t), \quad \eta_\delta(0) = 0.$$

and

$$\lim_{t\to\infty} \frac{\eta_\delta(t)}{t^{-\gamma\delta}} = 1. \tag{3.6}$$

Remark 3.8. Combining Proposition 3.1 and Theorem 3.5, we can conclude that when $\alpha > \max\{|a|, 1\}$, the initial value problem for (1.3) has infinitely many solutions that are globally decaying in time.

4 α-Riccati and Related Critical Phenomena

We start by defining several key notions describing the *time evolution* of the binary tree structure corresponding to the inhomogeneous Yule random field $\mathbf{Y}$ underlying the α-Riccati equation.

Note that the recursions (2.16) and (2.17) end at a vertex $v \in \mathbb{T}$ if and only if

$$\sum_{j=0}^{|v|-1} Y_{v|j} < t \leq \sum_{j=0}^{|v|} Y_{v|j} .$$

Recall, $Y_{v|j} = \alpha^{-j} T_{v|j}$, and thus can be viewed as branching clocks for the random field $\mathbf{Y}$.

As in the case of the unary tree, let $N(t) = |\overset{o}{V}(t)|$ (see (4.2) below) be the number of clock rings in the binary tree by time t. It was observed in passing in [1] and proven in [8] that for $\alpha = 1/2$, the distribution of $N(t) - 1$ is Poisson distributed. It follows that $\mathbf{X}(t)$ defined by (2.16) has a *log-Poisson distribution*. It is also to be noted that at least four interesting critical parameter ranges, $\alpha < 1, \alpha = 1, 1 < \alpha < \sqrt{2}, \alpha > \sqrt{2}$, were identified in [1] for distinct qualitative changes in behavior of the α-Riccati model. In fact the α-Riccati model has been shown to be a rich source of critical phenomena on both large (averaged) and small (stochastic) scales. In particular, specific critical regimes in the stochastic solution process that affect the averaging for solutions as expected values are the subject of this section.

With the convention that $\sum_{j=0}^{-1} = 0$, we define the continuous parameter Markov process of sets of "t-leaves" by

$$\partial V(t) = \left\{ v \in \mathbb{T} : \sum_{j=0}^{|v|-1} Y_{v|j} < t \leq \sum_{j=0}^{|v|} Y_{v|j} \right\}, \tag{4.1}$$

the corresponding set of ancestors by

$$\overset{o}{V}(t) = \left\{ u \in \mathbb{T} : \sum_{j=0}^{|u|} Y_{u|j} < t \right\}, \tag{4.2}$$

and

$$V(t) = \overset{o}{V}(t) \cup \partial V(t). \tag{4.3}$$

Up to an explosion time S defined by

$$S = \inf_{s \in \partial \mathbb{T}} \sum_{j=0}^{\infty} Y_{s|j}, \tag{4.4}$$

$V(t)$ takes values in a denumerable and partially ordered *evolutionary space* $\mathscr{E}$ of nonempty, finite, connected, rooted at θ subtrees[¶] of vertices of the binary tree $\mathbb{T}$. Specifically, $\mathscr{E}$ may be viewed inductively as consisting of finite trees $A \subset \mathbb{T}$ such that $A = \{\theta\}$, or there is a $B \in \mathscr{E}$ such that $A = B \cup \{v1, v2\}$ for some $v \in B$, with offspring $v1, v2 \notin B$. Upon explosion the subtrees are no longer finite nor binary, however with an extension to a space, say $\overline{\mathscr{E}}$, that permits infinite trees, the evolution of $V(t)$ naturally continues in $\overline{\mathscr{E}}$.

If one regards

$$\Theta_v = \sum_{j=0}^{|v|} Y_{v|j}, \; v \in \mathbb{T}, \quad \Theta_{\overleftarrow{\theta}} = 0, \tag{4.5}$$

as a *replacement time* of the vertex v, then $v \in \overset{o}{V}(t)$ iff v dies prior to time t, and $v \in \partial V(t)$ iff v lives beyond time t, but its parent $\overleftarrow{v}$ dies prior to t. Upon replacement, a vertex $v \in \mathbb{T}$ branches into two offspring $v1, v2$, respectively. So Θ_v is also referred to as the *branching time* of v. One may say that "$v \in \partial V(t)$ crosses t, while its parent $\overleftarrow{v}$ does not cross t".

One may also note that prior to *explosion* of $V(t)$ in the evolutionary space $\mathscr{E}$ at a possibly finite time $S < \infty$, $\partial V(t)$ is a finite set which evolves infinitesimly in time t to $t + dt$ by removal of a t-leaf $v \in \partial V(t)$ and replacement by its offspring $v1, v2$. On the other hand, after explosion the t-leaf process $\partial V(t)$ continues to evolve in a *canopy space* $\mathscr{C}$ of nonempty cutsets of (possibly infinite) subtrees induced by $\{Y_v : v \in \mathbb{T}\}$ rooted at θ. In particular, at any time after explosion of $\mathbf{Y}$, the t-leaf set $\partial V(t)$ remains well-defined, but may evolve to the empty set at possibly finite time

$$L = \sup_{s \in \partial\mathbb{T}} \sum_{j=0}^{\infty} Y_{s|j}, \tag{4.6}$$

an event referred to as *hyperexplosion*. In terms of the total time accumulated on a ray $s \in \partial\mathbb{T}$, $\Theta_s = \sum_{j=0}^{\infty} Y_{s|j}$, one may also write

$$S = \inf_{s \in \partial\mathbb{T}} \Theta_s, \quad L = \sup_{s \in \partial\mathbb{T}} \Theta_s. \tag{4.7}$$

The notations S and L for explosion and hyperexplosion times, respectively, are used to convey 'shortest' and 'longest' tree path lengths as measured by $\sum_{j=0}^{\infty} Y_{s|j}$, $s \in \partial\mathbb{T}$.

Remark 4.1. The event $[|\partial V(\ell, t)| = \infty]$ cannot be ruled out apriori since there are explosive trees for which this is possible (see Proposition 4.3 below).

In view of the binary tree structure, one has

$$|V(t)| = 2|\overset{o}{V}(t)| + 1, \;\; |\partial V(t)| = |\overset{o}{V}(t)| + 1 \quad t \geq 0, \tag{4.8}$$

where, according to (4.1), the singleton root $\{\theta\}$ counts as a t-leaf with *no* ancestors if $T_\theta > t$.

Theorem 4.2 (see [2, 14])**.** *The α-Riccati model is non-explosive for $\alpha \leq 1$ and hyperexplosive for $\alpha > 1$.*

[¶]A tree rooted at θ is a graph without loops and designated vertex θ as root. We identify such trees with their sets of vertices.

The following correction[‖] to Proposition 2.1 in [14] reveals a new critical phenomena of t-*leaf percolation* in addition to another solution to α-Riccati equations; namely the positive probability of infinitely many t-leaves iff $\alpha \in (1, 2)$.

Proposition 4.3 (t-Leaf Percolation, [13, 14])**.** *(i) For $\alpha \in [0, 1] \cup [2, \infty)$ one has $\mathbb{P}(|\partial V(t)| < \infty) = 1$ for all $t \geq 0$. (ii) For $1 < \alpha < 2$, one has $\mathbb{P}(|\partial V(t)| = \infty) > 0$ for all $t > 0$.*

More is actually true with regard to the distributions of the explosion time S and hyperexplosion time L for α-Riccati that improves on [2] as follows.

Proposition 4.4. *For $0 < \alpha \leq 1$, $S = L = \infty$ a.s.. For $\alpha > 1$, $\mathbb{P}(L > t) \leq \mathbb{P}(S > t) = O(e^{-\frac{\alpha-1}{\alpha}t})$ as $t \to \infty$. In particular, $\mathbb{E}(S) \leq \mathbb{E}(L) < \infty$.*

Proof. The case $0 < \alpha \leq 1$ follows from well-known properties of the standard Yule model. Assume $\alpha > 1$. Let $M_j = \alpha^{-j} \max\{T_1^{(j)}, \ldots, T_{2^j}^{(j)}\}$, where $T_i^{(j)}, i = 1, \ldots, 2^j, j = 1, 2, \ldots$, are i.i.d. mean one exponentially distributed random variables. Then,

$$L_n = \max_{|v|=n} \sum_{j=0}^{n} \alpha^{-j} T_{v|j} \overset{dist}{\leq} \sum_{j=0}^{n} M_j, \qquad L = \lim_n L_n \overset{dist}{\leq} \sum_{j=0}^{\infty} M_j.$$

Also,

$$\begin{aligned} \pi_j = \mathbb{P}(M_j > \delta_j) &= 1 - \mathbb{P}(M_j \leq \delta_j) \\ &= 1 - (1 - e^{-\delta_j \alpha^j})^{2^j} \\ &\leq e^{j \ln 2 - \delta_j \alpha^j}. \end{aligned} \tag{4.9}$$

Fix $t > 0$. Note that $\sum_{j=0}^{\infty} \pi_j \leq e^{-t}$ for δ_j selected such that

$$e^{j \ln 2 - \delta_j \alpha^j} \leq e^{-t} \frac{1}{c_0 (j+1)^2}, \quad c_0 = \pi^2/6 > 1,$$

i.e., for a choice of

$$\delta_j \geq \delta_j(t) = \frac{t + j \ln 2 + 2 \ln(j+1) + \ln c_0}{\alpha^j}, \quad j = 0, 1, \ldots.$$

Moreover,

$$\sum_{j=0}^{\infty} \delta_j(t) = \sum_{j=0}^{\infty} \alpha^{-j} t + \sum_{j=0}^{\infty} \frac{j \ln 2 + 2 \ln(j+1) + \ln c_0}{\alpha^j} = c_1 t + c_2,$$

where $c_1 = \sum_{j=0}^{\infty} \alpha^{-j} = \frac{\alpha}{\alpha - 1}$, and $c_2 = \sum_{j=0}^{\infty} \frac{j \ln 2 + 2 \ln(j+1) + \ln c_0}{\alpha^j}$.

Now, note that $L \geq c_1 t + c_2$ implies that for some n, $M_n > \delta_n(t)$, and thus

$$\mathbb{P}(L \geq c_1 t + c_2) \leq \sum_{j=0}^{\infty} \mathbb{P}(M_j > \delta_j(t)) = e^{-t}.$$

[‖]This correction is proven in the errata [13].

Setting $t' = c_1 t + c_2$, i.e., $t = (t' - c_2)/c_1$, one has

$$\mathbb{P}(L \geq t) \leq e^{-\frac{t-c_2}{c_1}}, \quad \forall t \geq 0,$$

and the remaining statements of the proposition easily follow. □

Theorem 4.5. *A precise exponential rate of convergence holds for $\alpha > 1$:*

$$\mathbb{P}(L \geq t) \sim e^{-t} \text{ as } t \to \infty, \tag{4.10}$$

which is the same decay rate as $\mathbb{P}(S \geq t)$. Moreover, if $u(t)$ is a solution to α-Riccati such that $u(t) \to 1$ as $t \to \infty$, then only one of the following is possible:

(a) $|u(t) - 1| \gtrsim t^{-\gamma}$, *where* $\gamma = \log_\alpha 2$.

(b) $|u(t) - 1| \sim e^{-t}$, *or*

(c) $u(t) = 1$ *for all all* $t \geq 0$.

In the above, $f(t) \gtrsim g(t)$ means that there exist $c, T > 0$ such that $g(t) \geq c_2 f(t)$ for all $t \geq T$ and $f \sim g$ means $f \gtrsim g$ and $g \gtrsim f$.

Remark 4.6. The case (a) is illustrated by a special solution of [2]. Case (b) is illustrated by $u(t) = \mathbb{P}(L > t)$, and case (c) by $u(t) \equiv 1$.

The proof of Theorem 4.5 rests on the following lemma which couples the initial data 0 and 1 through the inclusion-exclusion principle. Namely, by inclusion-exclusion, $w(t) = \mathbb{P}(L \leq t)$ solves (4.11) below with $w(0) = 1$.

Lemma 4.7. *Assume $\alpha > 1$ in the α-Riccati model. Suppose that $w(t)$, $t > 0$ solves*

$$w'(t) = -w(t) + 2w(\alpha t) - w^2(\alpha t), \quad t \geq 0, \tag{4.11}$$

and assume that $|w(t)| = O(e^{-\gamma t})$ as $t \to \infty$ for some $\gamma > 0$.

(i) *If $0 < \gamma < 1$ then $w(t) = O(e^{-t})$ as $t \to \infty$.*

(ii) *If $\gamma > 1$ then $w(t) \equiv 0$.*

Proof. Part (i) is proven by a *bootstrap* method, starting from $w(t) = O(e^{-\gamma t})$ by hypothesis, with $\gamma \in (0, 1)$. Then, $|2w(t) - w^2(t)| \leq ce^{-\gamma t}$ for large enough $c = c_\gamma$. Using (4.11), one has

$$-w(t) - c\, e^{-\alpha\gamma t} \leq w'(t) \leq -w(t) + c\, e^{-\alpha\gamma t}.$$

Integrating on $[t_0, t]$ one has

$$\begin{aligned} |w(t) - w(t_0)e^{-(t-t_0)}| &\leq e^{-t} \int_{t_0}^{t} e^s c\, e^{-\alpha\gamma s} ds \\ &= \frac{c}{1 - \alpha\gamma} e^{-t} \left(e^{(1-\gamma\alpha)t} - e^{(1-\gamma\alpha)t_0}\right) \end{aligned}$$

so that as $t \to \infty$

$$|w(t)| = \begin{cases} O(e^{-t}) & \text{if } \alpha\gamma > 1 \\ O(e^{-\alpha\gamma t}) & \text{if } \alpha\gamma < 1. \end{cases}$$

In the case $\alpha\gamma > 1$, the process stops and (i) is established, while in the case $\alpha\gamma < 1$, the bootstrap process is repeated with γ replaced by $\alpha\gamma (> \gamma)$. Note that each time the bootstrap process is applied, another factor of α appears in the exponent. Thus, after k steps with $\alpha^k\gamma < 1$,

$$w(t) = O(e^{-\alpha^k \gamma t}).$$

Now, for $k \geq \frac{\ln(\frac{1}{\gamma})}{\ln(\alpha)}$ the process stops and $w(t) \leq O(e^{-t})$ is achieved.

To prove part (ii), assume that $|w(t)| = O(e^{-\gamma t})$, with $\gamma > 1$. By the same argument as above, for $c > 0$ big enough and $t > t_0 > 0$,

$$|w(t) - w(t_0)e^{-(t-t_0)}| \leq \frac{c}{1-\alpha\gamma} e^{-t} \left(e^{(1-\gamma\alpha)t} - e^{(1-\gamma\alpha)t_0} \right) \tag{4.12}$$

Note that if $w(t_0) \neq 0$, then $|w(t) - w(t_0)e^{-(t-t_0)}| \geq O(e^{-t})$, while $e^{-t} \left(e^{(1-\gamma\alpha)t} - e^{(1-\gamma\alpha)t_0} \right) = O(e^{-\gamma\alpha t}) = o(e^{-t})$, contradicting (4.12). This contradiction implies that $w(t_0) = 0$ for all $t_0 > 0$, i.e. $w(t) \equiv 0$. □

Proof of Theorem 4.5. Let $w(t) = 1 - u(t)$, then w satisfies (4.11). Assume $w(t) = o(t^{-\gamma})$, Note that (4.11) implies that

$$w'(t) = -w(t) + (2 - w(\alpha t))w(\alpha t),$$

and thus

$$w(t) = w(t_0)\, e^{-(t-t_0)} + \int_{t_0}^{t} (2 - w(\alpha s))\, e^{-(t-s)} w(\alpha s)\, ds. \tag{4.13}$$

As in the proof of Theorem 9(ii) in [18], it then follows that $|w(t)| = O(e^{-\epsilon t})$ for some $\epsilon > 0$. For completeness, we will present this argument below.

For $\tau > 0$ consider

$$m(\tau) = \sup_{t \geq \tau} t^\gamma |w(t)|.$$

Note that $m(\tau)$ is a bounded decreasing to zero function. Moreover, from (4.13)

$$|w(t)| \leq t_0^{-\gamma} e^{-(t-t_0)} m(t_0) + m(\alpha t_0) e^{-t} \int_{t_0}^{t} (2 + |w(\alpha s)|)\, e^s (\alpha s)^{-\gamma} ds.$$

Let $b \in (0,1)$ be such that $\nu = b - \alpha^{1/2} > 0$. Fix a $\tau_0 > 1$ big enough that $|w(t)| \leq t^{-\gamma}$ and $t^\gamma e^{-t} \leq e^{-bt}$ for all $t > \tau_0$. Then, for $t > t_0 > \tau_0$ we have

$$t^\gamma |w(t)| = t_0^{-\gamma} e^{-bt + t_0} + m(\alpha t_0)\, t^\gamma e^{-t} \int_{t_0}^{t} e^s \,(2 + (\alpha s)^{-\gamma})(\alpha s)^{-\gamma} ds.$$

Using the estimate $\int_1^t t^k e^s ds \le (1 + \frac{c_k}{t})\, e^t t^k$ (valid for any $k \in \mathbb{R}$ and big enough $c_k > 0$), we conclude

$$
\begin{aligned}
t^\gamma |w(t)| &= t_0^{-\gamma} e^{-bt+t_0} + m(\alpha t_0) \left(2\alpha^{-\gamma} \left(1 + \frac{c_\gamma}{t}\right) + \alpha^{-2\gamma} \left(1 + \frac{c_{2\gamma}}{t}\right) t^{-\gamma} \right) \\
&\le t_0^{-\gamma} e^{-bt+t_0} + m(\alpha t_0) \left(1 + \frac{C}{t^\delta}\right) .
\end{aligned}
$$

where $\delta = \min\{1, \gamma\}$, and $C > c_\gamma$ big enough, independent of t. In the above, we used the fact that $2\alpha^{-\gamma} = 1$. Consider $\tau = \alpha^{1/2} t_0$ and take $\sup_{t \ge \tau}$ in the right-hand side of the inequality above. We obtain:

$$
m(\tau) \le t_0^{-\gamma} e^{-b\tau + \alpha^{-1/2}\tau} + m(\alpha^{1/2}\tau) \left(1 + \frac{C}{\tau^\delta}\right) = e^{-\nu\tau} + \left(1 + \frac{C}{\tau^\delta}\right) m(\alpha^{1/2}\tau) . \tag{4.14}
$$

We can iterate (4.14), by applying it to $m(\alpha^{1/2}\tau)$ in the right-hand-side (with τ replaced with $\alpha^{1/2}\tau$), obtaining:

$$
\begin{aligned}
m(\tau) &\le e^{-\epsilon\tau} + \left(1 + \frac{C}{\tau^\delta}\right) \left(e^{-\nu\alpha^{1/2}\tau} + \left(1 + \frac{C}{(\alpha^{1/2}\tau)^\delta}\right) m(\alpha^{2/2}\tau) \right) \\
&\le e^{-\nu\tau} + \left(1 + \frac{C}{\tau^\delta}\right) e^{-\nu\alpha^{1/2}\tau} + \left(1 + \frac{C}{\tau^\delta}\right) \left(1 + \frac{C}{\alpha^{\delta/2}\tau^\delta}\right) m(\alpha^{2/2}\tau) \\
&\le \left(1 + \frac{C}{\tau^\delta}\right) \left(1 + \frac{C}{\alpha^{\delta/2}\tau^\delta}\right) \left(e^{-\nu\tau} + e^{-\nu\alpha^{1/2}\tau} + m(\alpha^{2/2}\tau) \right) \\
&\le e^{\left(1 + \frac{1}{\alpha^{\delta/2}}\right) \frac{C}{\tau^\delta}} \left(e^{-\nu\tau} + e^{-\nu\alpha^{1/2}\tau} + m(\alpha^{2/2}\tau) \right) .
\end{aligned}
$$

Iterating this process n times, we estimate:

$$
\begin{aligned}
m(\tau) &\le e^{\sum_{j=0}^{n-1} \frac{1}{\alpha^{j\delta/2}} \frac{C}{\tau}} \left(\sum_{j=0}^{n-1} e^{-\nu\alpha^{j/2}\tau} + m(\alpha^{n/2}\tau) \right) \\
&\le e^{\frac{\alpha^{\delta/2}}{\alpha^{\delta/2}-1} \frac{C}{\tau}} \left(\sum_{j=0}^{n-1} e^{-\nu\alpha^{j/2}\tau} + m(\alpha^{n/2}\tau) \right) .
\end{aligned}
$$

Now take $n \to \infty$, using that $m(t) \to 0$ as $t \to \infty$ to obtain

$$
m(\tau) \le e^{\frac{\alpha^{\delta/2}}{\alpha^{\delta/2}-1} \frac{C}{\tau}} \sum_{j=0}^{\infty} e^{-\nu\alpha^{j/2}\tau} ,
$$

and consequently, $m(\tau) = O(e^{\nu\tau})$. Thus, $w(t) = o(e^{\epsilon t})$ with $\epsilon = \nu/2$. Then, the conclusions of Theorem 4.5 follow from Lemma 4.7. □

The fine scale structure of the α-Riccati model can be further delineated in showing that every explosive by time $t > 0$ tree has a hyperexplosive subtree.

Proposition 4.8. *Suppose $\alpha > 1$. Then, writing $v \in s$ to mean $v = s|m$ for some $m \geq 0$, let*

$$L_v = \sup_{s\in\partial\mathbb{T}, v\in s} \sum_{j=|v|}^{\infty} \alpha^{-j} T_{s|j} \overset{dist}{=} \alpha^{-|v|} L_\theta, \quad v \in \mathbb{T}.$$

Then,

$$P(\cup_{v\in\mathbb{T}}[L_v \leq t - \Theta_v] \mid S{\leq}t) = 1.$$

Proof. Recall the notation for the branching time of $v \in \mathbb{T}$ defined by (4.5), i.e.,

$$\Theta_s = \sum_{j=0}^{\infty} \alpha^{-j} T_{s|j}, \ s \in \partial\mathbb{T}, \qquad \Theta_v = \sum_{j=0}^{|v|} \alpha^{-j} T_{v|j}, \ v \in \mathbb{T}. \tag{4.15}$$

Fix $t > 0$. Let $H_t = \cup_{v\in\mathbb{T}}[L_v \leq t - \Theta_v] \in \mathscr{F}$ denote the event that there is a hyper-explosive subtree by time t. Observe that one may bound the conditional probability that there is no hyper-explosion at v given the event $[t - \Theta_v \leq \frac{1}{n}]$ as follows:

$$\begin{aligned}
\mathbb{P}(L_v > t - \Theta_v | \, t - \Theta_v {\geq} \frac{1}{n}) &\leq \mathbb{P}(L_v > \frac{1}{n} \,|\, t - \Theta_v {\geq} \frac{1}{n}) \\
&= \mathbb{P}(L_v > \frac{1}{n}) = \mathbb{P}(L_\theta > \frac{\alpha^{|v|}}{n}) \leq c\exp(-\frac{\alpha^{|v|}}{n}),
\end{aligned} \tag{4.16}$$

since by Theorem 4.5, $\mathbb{P}(L_\theta > r) \leq ce^{-r}, r \geq 0$, for some constant $c > 0$. Now, choose $K_n > n$, such that

$$c\exp(-\frac{\alpha^m}{n}) < 3^{-m}, \quad \forall m \geq K_n.$$

This is possible since for every $n \geq 1$, $3^m \exp(-\frac{\alpha^m}{n}) = o(1)$ as $m \to \infty$. Thus, by (4.16) we have

$$\mathbb{P}([L_v > t - \Theta_v] \cap [t - \Theta_v {\geq} \frac{1}{n}]) \leq 3^{-|v|} \quad \text{for all } v \in \mathbb{T} \text{ with } |v| = K_n.$$

Observe that for any $N \geq 1$,

$$[S{\leq}t] = \cup_{n\geq N} \cup_{|v|=K_n, v\in\mathbb{T}} \left(\cup_{s\in\partial\mathbb{T}, v\in s}[\Theta_s \leq t - \frac{1}{n}] \right). \tag{4.17}$$

To see measurability of $\cup_{s\in\partial\mathbb{T}, v\in s}[\Theta_s \leq t - \frac{1}{n}]$, observe that for a fixed $v \in \mathbb{T}$

$$\cup_{s\in\partial\mathbb{T}, v\in s}[\Theta_s \leq t - \frac{1}{n}] = \cap_{m\geq|v|}^{\infty} \cup_{u\in\mathbb{T}, v\in u, |u|=m} [\Theta_u \leq t - \frac{1}{n}].$$

Thus, for arbitrary $n \geq 1$, since $H_t^c = \cap_{j=1}^{\infty} \cap_{|v|=j} [L_v > t - \Theta_v] \subseteq \cap_{|v|=K_n}[L_v > t - \Theta_v]$, and since for $v \in s \in \partial\mathbb{T}, [t - \Theta_s \geq 1/n] \subseteq [t - \Theta_v \geq 1/n]$, one has

$$H_t^c \cap [S{\leq}t] \subseteq \cup_{n\geq N} \cup_{|v|=K_n} ([L_v > t - \Theta_v] \cap [t - \Theta_v {\geq} \frac{1}{n}]).$$

Therefore, using $K_n > n$, for all $n \geq N$

$$\begin{aligned}\mathbb{P}(H_t^c \cap [S \leq t]) &\leq \sum_{n\geq N}\sum_{|v|=K_n} \mathbb{P}([L_v > t-\Theta_v] \cap [t-\Theta_v \geq \frac{1}{n}]) \\ &\leq \sum_{n\geq N}\sum_{|v|=K_n} 3^{-|v|} = \sum_{n\geq N} 2^{K_n}3^{-K_n} \leq \sum_{n\geq N}\left(\frac{2}{3}\right)^n = 3\left(\frac{2}{3}\right)^N.\end{aligned}$$

Thus, letting $N \to \infty$,

$$\mathbb{P}(H_t^c \cap [S \leq t]) = 0.$$

Since $0 < \mathbb{P}(S \leq t) < 1$, the assertion follows. □

In the next section, we will use Proposition 4.8 in the context of the stochastic Picard ground state iteration method, described in Section 5, (see Proposition 5.4 and Lemma 5.6).

5 The Solution Process, Mean Flow Equation, and Stochastic Picard Ground State Iterations

The purpose of the present section is to briefly review the stochastic Picard ground state method in the context of well-posedness problems for (2.6). The methods of [2] provide a useful view of the solution process $\mathbf{X}$ from the perspective of extreme value theory [2, 14, 22]. Specifically, the stochastic recursion expresses the length of the *longest* ray-indexed sum as the (independent) sum of Y_θ plus the maximum of the longest paths of the two subtrees emerging from vertices 1 and 2; the mean of which is (1.2). Such recursive structure inspired the stochastic Picard ground state method for constructing solution processes in [7] by the iterative methods described below.

In [14], the following idea, inspired by the proof of the uniqueness results for the Navier-Stokes equations in [19], provides an iterative approach for proving non-uniqueness of mean flow equations in the context of stochastic explosion. This method will be referred to as the *stochastic Picard ground state iterations*, or simply *stochastic Picard iterations*. This method is suitable for non-linear systems, such a (2.6), in contrast to a more typical probabilistic approach to linear parabolic equations, and homogeneous Markov processes in general, where explosion of the associated Markov process can be exploited for non-uniqueness by re-initiating the process at the time of explosion S to construct distinct distributions with the same local behavior. As shown below, for the explosive linear pantograph equation (2.5) ($\alpha > 1$), the minimal solution of (1.3) in the context of stochastic Picard iterations approach coincides with that obtained from the standard (Feller) Markov process theory minimal solution. While the latter is not applicable to the *non-linear* α-Riccati equation, the former does yield solutions.

In the present context of (2.6), the stochastic Picard iterations proceed by considering an arbitrary initial "ground state" process $X_0(t)$ to be determined. Define $X_n(t)$ sample pointwise by

$$X_n(t) = \begin{cases} u_0 & \text{if } T_\theta > t \\ X_{n-1}^{(1)}(\alpha(t-T_\theta))\, X_{n-1}^{(2)}(\alpha(t-T_\theta)) & \text{otherwise,} \end{cases} \tag{5.1}$$

Where $X_{n-1}^{(1)}$ and $X_{n-1}^{(2)}$ are conditionally on T_θ independent copies of X_{n-1}, and, as in (2.16), T_θ the root clock distributed exponentially with intensity 1. Thus, $X_n(t)$ is a finite product where each t-leaf v, with $|v| < n$, contributes u_0 and each v, with $|v| = n$ (the truncated branch), contributes $X_0(\alpha^{|v|}(t-\Theta_v))$. Note that, since $X_0(t)$ is a stochastic process, by induction, for any $n \in \mathbb{N}$, $X_n(t)$ is a well-defined progressively measured stochastic process. Moreover, if $u^{(0)}(t) = \mathbb{E}(X_0(t))$ is well-defined, then the sequence $u^{(n)} = \mathbb{E}(X_n), n \geq 0$ is well-defined and formally satisfies Picard-type iterations of (2.6):

$$u^{(n)}(t) \;=\; u_0 e^{-t} + \int_0^t e^{-s}[u^{(n-1)}(\alpha(t-s))]^2 ds \tag{5.2}$$

We have the following result about the convergence of the stochastic iterations to a solutions process.

Theorem 5.1. *Let $X_n(t)$ be the sequence of stochastic Picard iterations satisfying* (5.1)*. Suppose that for all $t > 0$, $X_n(t)$ is convergent a.s. as $n \to \infty$. Then there exists a stochastic process $X(t)$ such that $X_n(t,\omega) \to X(t,\omega)$ a.e. on $[0,\infty) \times \Omega$ with respect to the product measure $\mu_{[0,\infty)} \otimes \mathbb{P}$ as $n \to \infty$, where $\mu_{[0,\infty)}$ is a Borel measure on $[0,\infty)$. Moreover, for all $t > 0$, $X(t)$ is a solution process for the α-Riccati equation, satisfying* (2.16) *a.s.*

Proof. We model the probability space by $\overline{\omega} = (\omega_v)_{v\in\mathbb{T}} \in \Omega = [0,\infty)^{\mathbb{T}}$, with $(\Omega, \mathscr{B}_\Omega, \mathbb{P})$ being a product probability space of countably many intensity one probability measures exponential measures defined on $\mathscr{B}_{[0,\infty)}$ – the Borel σ-algebras of $[0,\infty)$: $d\mathbb{P} = \prod_{v\in\mathbb{T}} d\mathbb{P}_v$, with $d\mathbb{P}_v(\omega_v) = e^{-\omega_v} d\omega_v$. Thus, in this setting, the exponential clocks are $T_v(\overline{\omega}) = \omega_v$.

Let $\mathbb{T}_1 = \{v \in \mathbb{T} \setminus \{\theta\} :\ v|1 = 1\}$ and $\mathbb{T}_2 = \{v \in \mathbb{T} \setminus \{\theta\} :\ v|1 = 2\}$ be the "left" and right subtrees of $\mathbb{T}$. Write $\overline{\omega}^{(1)} = (\omega_v)_{v\in\mathbb{T}_1} \in \Omega_1 = [0,\infty)^{\mathbb{T}_1}$ and $\overline{\omega}^{(2)} = (\omega_v)_{v\in\mathbb{T}_2} \in \Omega_2 = [0,\infty)^{\mathbb{T}_2}$. Thus, we can view $(\Omega, \mathbb{P})$ as a product space: $\overline{\omega} = (\omega_0, \overline{\omega}^{(1)}, \overline{\omega}^{(2)}) \in \Omega = [0,\infty) \times \Omega_1 \times \Omega_2$ and $d\mathbb{P}(\overline{\omega}) = d\mathbb{P}_\theta(\omega_\theta) \otimes d\mathbb{P}^{(1)}(\overline{\omega}^{(1)}) \otimes d\mathbb{P}^{(2)}(\overline{\omega}^{(2)})$ with $d\mathbb{P}_\theta(\omega_\theta) = e^{-\omega_\theta} d\omega_\theta$, and $d\mathbb{P}^{(j)}(\overline{\omega}^{(j)}) = \prod_{v\in\mathbb{T}_j} e^{-\omega_v} d\omega_v$, $j = 1,2$. Note that $(\Omega,\mathbb{P})$, $(\Omega_j, \mathbb{P}^{(j)})$, $j = 1,2$ are identically distributed probability spaces.

Let $\mathscr{E} = \{(t,\overline{\omega}) \in [0,\infty) \times \Omega :\ X_n(t,\omega) \text{ is convergent in } \mathbb{R}\}$. Since X_n is measurable in t and $\overline{\omega}$, $\mathscr{E}$ is measurable with respect to the σ-algebra $\mathscr{B}_{[0,\infty)} \otimes \mathscr{B}_\Omega$. Note that since for any $t > 0$ $X_n(t)$ is convergent a.s., $\mathbb{P}(\mathscr{E}_t) = 1$, where $\mathscr{E}_t = \{\overline{\omega} \in \Omega |\ (t,\overline{\omega}) \in \mathscr{E}\}$. Thus, by the Fubini's Theorem,

$$[\mu_{[0,\infty)} \otimes \mathbb{P}](\mathscr{E}^c) = \int_0^\infty \mathbb{P}(\Omega \setminus \mathscr{E}_t)\, dt = 0.$$

Define

$$X(t,\overline{\omega}) = \begin{cases} \lim\limits_{n\to\infty} X_n(t,\overline{\omega}), & (t,\overline{\omega}) \in A; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $X(t)$ is a well-defined progressively measured stochastic process and $X_n \to X$ a.e. in $(t,\overline{\omega})$.

To show $X(t)$ is a solution process, fix a $t > 0$. Suppose $T_\theta = \omega_\theta < t$. In this setting, in (5.1) and (2.16), $X_{n-1}^{(j)}(\alpha(t - T_\theta)) = X_{n-1}(\alpha(t-\omega_\theta), \overline{\omega}^{(j)})$, and $X^{(j)}(\alpha(t-T_\theta)) = X(\alpha(t-\omega_\theta), \overline{\omega}^{(j)})$, $j = 1,2$. Note that for all $n \in \mathbb{N}$ and $j = 1,2$ $X_n^{(j)} \mathbb{1}_{[\omega_\theta < t]}$ is measurable in $(t,\overline{\omega})$. For $j = 1,2$, let

$$\mathscr{E}_t^{(j)}(\omega_\theta) = \left\{\overline{\omega}^{(j)} \in \Omega_j \ :\ \omega_\theta < t, \text{ and } X_{n-1}(\alpha(t-\omega_\theta), \overline{\omega}^{(j)}) \to X(\alpha(t-\omega_\theta), \overline{\omega}^{(j)}) \text{ as } n \to \infty\right\}.$$

Since $X_n \to X$ a.s.,

$$\mathbb{P}^{(j)}(\mathscr{E}_t^{(j)}(\omega_\theta)) = \int_{\Omega_j} \mathbb{1}_{\mathscr{E}^{(j)}(\omega_\theta)}(\overline{\omega}^{(j)})\, d\mathbb{P}^{(j)}(\overline{\omega}^{(j)}) = 1, \quad j = 1, 2,\ \omega_\theta < t.$$

Let

$$\tilde{\mathscr{E}}_t = \left\{ \overline{\omega} = (\omega_\theta, \overline{\omega}^{(1)}, \overline{\omega}^{(2)}) \;:\; \omega_\theta \geq t \text{ or } [\omega_\theta < t \text{ and } \overline{\omega}^{(j)} \in \mathscr{E}_t^{(j)}(\omega_\theta),\ j = 1, 2] \right\} .$$

Note that we have

$$\begin{aligned} \tilde{\mathscr{E}}_t =& \{\overline{\omega} \;:\; T_\theta(\overline{\omega}) \geq t\} \\ & \cup \{\overline{\omega} \;:\; T_\theta(\overline{\omega}) < t \text{ and } X_n(\alpha(t - T_\theta(\overline{\omega})), \overline{\omega}^{(j)}) \to X(\alpha(t - T_\theta(\overline{\omega})), \overline{\omega}^{(j)}),\ j = 1, 2\}. \end{aligned}$$

Since $X_n(\cdot, \cdot)$, $T_\theta(\cdot)$ as well as the the mappings $(t, \overline{\omega}) \to (\alpha(t - \omega_\theta), \overline{\omega}^{(i)})$, $j = 1, 2$, are measurable, we conclude that $\tilde{\mathscr{E}}_t$ is measurable. Thus, by Fubini's Theorem

$$\begin{aligned} \mathbb{P}(\tilde{\mathscr{E}}_t) &= e^{-t} + \int_0^t e^{-\omega_\theta} \int_{\Omega_1} \mathbb{1}_{\mathscr{E}_t^{(1)}(\omega_\theta)}(\overline{\omega}^{(1)})\, d\mathbb{P}^{(1)}(\overline{\omega}^{(1)}) \int_{\Omega_2} \mathbb{1}_{\mathscr{E}_t^{(2)}(\omega_\theta)}(\overline{\omega}^{(2)})\, d\mathbb{P}^{(2)}(\overline{\omega}^{(2)})\, d\omega_\theta \\ &= e^{-t} + \int_0^t e^{-\omega_\theta}\, 1 \cdot 1\, d\omega_\theta = 1\, . \end{aligned}$$

The proof is finished once we observe that for any $\overline{\omega} \in \tilde{\mathscr{E}}_t$, (2.16) follows from (5.1) by taking $n \to \infty$. ☐

By a straightforward adaptation of the proof above, we can prove analogous convergence results for the stochastic Picard iterations corresponding to the pantograph equation (1.3) in both binary tree representation (2.17) and the unary tree representation (2.10).

Corollary 5.2. *Let $\mathcal{X}_n(t)$ be the sequence of binary cascade stochastic Picard iterations for* (1.3)

$$\mathcal{X}_n(t) = \begin{cases} u_0 & \text{if} \quad T_\theta \geq t, \\ \frac{a}{2}\mathcal{X}_{n-1}^{(1)}(\alpha(t - T_\theta)) + \frac{a}{2}\mathcal{X}_{n-1}^{(2)}(\alpha(t - T_\theta)) & \text{if} \quad T_\theta < t. \end{cases} \tag{5.3}$$

and $\tilde{\mathcal{X}}_n(t)$ is the sequence of unary cascade stochastic Picard iterations for (1.3)*:*

$$\tilde{\mathcal{X}}_n(t) = \begin{cases} u_0 & \text{if} \quad T_0 \geq t, \\ a\mathcal{X}_{n-1}^{(1)}(\alpha(t - T_0)) & \text{if} \quad T_0 < t. \end{cases} \tag{5.4}$$

Suppose that for all $t > 0$, $\mathcal{X}_n(t)$ is convergent a.s. as $n \to \infty$. Then there exists a stochastic process $\mathcal{X}(t)$ such that $\mathcal{X}_n(t, \omega) \to \mathcal{X}(t, \omega)$ as $n \to \infty$ a.e. on $[0, \infty) \times \Omega$ with respect to the product measure $\mu_{[0,\infty)} \otimes \mathbb{P}$, where $\mu_{[0,\infty)}$ is a Borel measure on $[0, \infty)$. Moreover, for all $t > 0$, $\mathcal{X}(t)$ is a binary solution process for the pantograph equation, satisfying (2.17) *a.s.*

Similarly, if for all $t > 0$, $\tilde{\mathcal{X}}_n(t)$ is convergent a.s. as $n \to \infty$,then there exists a stochastic process $\tilde{\mathcal{X}}(t)$ such that $\tilde{\mathcal{X}}_n(t, \omega) \to \tilde{\mathcal{X}}(t, \omega)$ as $n \to \infty$ a.e. on $[0, \infty) \times \Omega$ with respect to the product measure $\mu_{[0,\infty)} \otimes \mathbb{P}$, where $\mu_{[0,\infty)}$ is a Borel measure on $[0, \infty)$. Moreover, for all $t > 0$, $\tilde{\mathcal{X}}(t)$ is a unary solution process for the pantograph equation, satisfying (2.10) *a.s.*

If the explosion time $S > t$, then $X_n(t)$ is an eventual constant sequence, equal to $X(t)$ satisfying (2.16) for big enough n. In the explosive case, i.e. when $\alpha > 1$, $\mathbb{P}(S < \infty) = 1$, different choices of the ground state X_0, led to super-martingales yielding in the limit to multiple solutions for the same initial states u_0, [14]. Notably, for the initial condition $u_0 = 0$, the choice of a random initial iteration,

$$X_0(t) = \begin{cases} 0, & T_\theta \geq t, \\ G(t - T_\theta), & T_\theta < t, \end{cases} \tag{5.5}$$

where G is a continuous function, leads to a uniformly integrable super-martingale $\{X_n\}$, provided $u^{(0)} = \mathbb{E}(X_0)$ satisfies $[u^{(0)}(\alpha t)]^2 \leq G(t)$. If one chooses $G(t) \in [0, 1]$ (e.g. $G(t) \equiv 0$ or $G \equiv 1$), one obtains a uniformly integrable super-martingale, yielding in the limit of expectations solutions for the α-Riccati equation (1.2). One remarkable choice of G used in [14] is

$$G_A(t) = e^{-t^{-\gamma}}(1 + \gamma t^{-(\gamma+1)}), \quad \gamma = \frac{\ln 2}{\ln \alpha},$$

which yields a solution obtained earlier by Athreya [2] using an extreme value method. Notably, this special choice of ground state G_A is implicitly connected to the Frechet extreme value distribution with parameter γ.

The *minimal solution process* $\underline{X}(t)$ extends $X(t)$ past explosion time by setting it equal to 0. Alternatively, $\underline{X}$ is the limit, as $n \to \infty$ of the iterative process X_n, described above, corresponding to the ground state $X_0 \equiv 0$. It is easy to verify that the minimal process satisfies (5.1) for all $t > 0$.

In the non-explosive case, we have the following existence and uniqueness results connected to stochastic Picard iterations, [9, 14].

Proposition 5.3. *Let $\alpha \in (0, 1]$. Then, for any choice of ground state, X_0,*

$$X_n(t) \to \underline{X}(t) = u_0^{N_t} \qquad \textit{for all } t \geq 0,$$

where $N_t = |\partial V(t)| < \infty$ a.s.. Moreover:

1. *If $\alpha \in (0, 1)$, then $\underline{u}(t) = \mathbb{E}(\underline{X}(t)) < \infty$ for all $u_0, t > 0$ and, as $t \to \infty$*

$$\underline{u}(t) \to \begin{cases} 0, & \textit{if } u_0 \in [0, 1); \\ \infty, & \textit{if } u_0 > 1. \end{cases}$$

   ($\underline{u}(t) \equiv 1$ if $u_0 = 1$).

2. *If $\alpha = 1$, then $\underline{u}(t) = \mathbb{E}(\underline{X}(t)) < \infty$ solves the logistic equation $u' = -u + u^2$ with corresponding asymptotic behavior in t.*

In the explosive case, the following non-uniqueness results involving the use of constant ground states goes back to [14].

Proposition 5.4. *Let $\alpha > 1$. Consider the stochastic Picard ground state iterations $X_n(t)$ for (2.6) with constant ground states $X_0(t) \equiv \delta > 0$. As before, denote $N_t = |\partial V(t)|$ Then,*

(1) If $u_0 \in [0, 1]$ then

(i) *For* $\delta \in (0,1)$*,*

$$X_n \to \underline{X} = \begin{cases} 0 & \text{if } S < t \\ u_0^{N_t}, & \text{if } S \geq t. \end{cases}$$

In particular, the minimal solution $u(t) = \underline{u}(t) = \mathbb{E}(\underline{X})$ *is well-defined and* $\underline{u}(t) \to 0$ *as* $t \to \infty$*.*

(ii) *For* $\delta = 1$*,*

$$X_n \to \overline{X} = \begin{cases} 1 & \text{if } L < t \\ u_0^{N_t}, & \text{if } L \geq t. \end{cases}$$

In particular, $u(t) = \overline{u}(t) = \mathbb{E}(\overline{X})$ *is well-defined and* $\overline{u}(t) \to 1$ *as* $t \to \infty$

(iii) *For* $\delta > 1$*, the limit*

$$X_\infty(t) = \lim_{n\to\infty} X_n(t) = \begin{cases} \infty & \text{if } S < t \\ u_0^{N_t}, & \text{if } S \geq t. \end{cases}$$

In particular, $\mathbb{E}(X_\infty(t)) = \infty$ *for all* $t > 0$

(2) *Suppose that* $u_0 > 1$*.*

(i) *For* $\delta \in [0,1)$*,*

$$X_n \to \overline{X} = \begin{cases} 0, & \text{if } S < t \\ u_0^{N_t}, & \text{if } S \geq t. \end{cases}$$

In particular, for all $t > 0$

$$\underline{u}(t) = \mathbb{E}(\underline{X}(t)) \begin{cases} < \infty, & \text{if } u_0 < (2\alpha-1)/4, \\ = \infty, & \text{in finite time if } u_0 > 2\alpha - 1, \\ \text{unknown} & \text{in other cases.} \end{cases}$$

If there is a locally integrable function g *such that* $X_n \leq g$ *for all* n*, then* $\underline{u}(t) = \mathbb{E}\underline{X}(t) < \infty$*.*

(ii) *For* $\delta = 1$*,*

$$X_n(t) \to \overline{X}(t) = \begin{cases} \infty & \text{on } [N_t = \infty] \\ u_0^{N_t} & \text{on } [N_t < \infty]. \end{cases}$$

In particular, if $\alpha \in (1,2)$*, then* $\mathbb{E}(\overline{X}(t)) = \infty$*, and if* $\alpha > 2$ *then* $\overline{X}(t) \in \mathbb{R}$ *is well-defined for all* $t > 0$*, while*

$$\overline{u}(t) = \mathbb{E}(\overline{X}(t)) \begin{cases} < \infty, & \text{if } u_0 < (2\alpha-1)/4 - (6\alpha^2 - 15\alpha + 4)/4(\alpha-1)(2\alpha-1), \\ = \infty, & \text{in finite time if } u_0 > 2\alpha - 1 \text{ or } \alpha \in (0,2), \\ \text{unknown} & \text{in other cases.} \end{cases}$$

(iii) For $\delta > 1$,

$$X_n \to X_\infty = \begin{cases} \infty & \text{if } S < t \\ u_0^{N_t}, & \text{if } S \geq t. \end{cases}$$

$u(t) = \mathbb{E}(X_\infty(t)) = \infty$ *for all* $t > 0$.

Proof. In all cases, the particular form of limits of X_n follow from the explicit representation (5.6) and Lemma 5.6 below. The statements about the expectations are proven in [14], with additional input from Proposition 4.3 in the case $\alpha \in (1,2)$. Namely, the cases in part (1) follow from [14, Section 4, Proposition 4.1 and Theorem 4.1]. The case 2(i) is proven in [14, Theorems 3.3 and 5.1]. The part (2)(ii) is proven in [14, Theorems 4.2 and 5.1] (noting that $\overline{u}(t) \geq \underline{u}(t)$); moreover, Proposition 4.3 is used to conclude infiniteness of the expectation in the case $\alpha \in (1,2)$. Finally, the case (2)(iii) follows from the fact that $\mathbb{P}(S < t) > 0$. □

In preparation for Lemma 5.6, it is convenient to introduce an alternative representation of $\partial\mathbb{T}$. Identify $\partial\mathbb{T} = \{1,2\}^\infty$ with points in the unit interval under dyadic expansion. In particular, the ray $s = (s_1, s_2, \dots) \in \partial\mathbb{T}$ defines $x_s = \sum_{j=1}^{\infty}(s_j - 1)2^{-j} \in [0,1]$. Then, for $v \in \mathbb{T}$, the set of rays passing through v define a subinterval $J_v = [\sum_{j=1}^{|v|}(v_j-1)2^{-j}, \sum_{j=1}^{|v|}(v_j-1)2^{-j} + 2^{-|v|}]$. The countable set of rationals in $[0,1]$ admit two dyadic representations as rays.

Definition 5.5. A hyperexplosive subtree is said to be maximal if it is not a proper subtree of a larger hyperexplosive subtree.

Note that a maximal hyperexplosive subtree rooted at $v \in \mathbb{T}$ corresponds to rays belonging to the interval

$$J_v = [x_v, x_v + 2^{-|v|}], \quad x_v = \sum_{j=1}^{|v|}(v_j - 1)2^{-j}.$$

Lemma 5.6. *Let $X_0 = \delta$ and let*

$$M_n(t) = |\{v \in \overset{o}{V}(t) : |v| = n\}|,$$

and

$$N_n(t) = |\{v \in \partial V(t) : |v| \leq n\}|.$$

Then, the Picard ground state iteration at generation n is given by

$$X_n(t) = u_0^{N_n(t)}\delta^{M_n(t)}. \tag{5.6}$$

On the event $[S < t]$ one has:

(i) $u_0 > 1, \delta \in [0,1)$ implies $X_n(t) \to 0$ as $n \to \infty$.

(ii) $u_0 \in (0,1), \delta > 1$ implies $X_n(t) \to \infty$ as $n \to \infty$.

Proof. The explicit representation (5.6) directly from (5.1) by induction.

Now assume the event $[S < t]$. Not that in this case, $M_n(t) > 0$ for all $t > 0$, so by if $\delta = 0$, (5.6), $X_n(t) = 0 \to 0$ as $n \to \infty$. Thus, it remains to consider the case $\delta > 0$.

By Proposition 4.8, $V(t)$ contains maximal hypexplosive subtrees. In accordance with the Definition 5.5, at generation $n > |v|$, a maximal hyperexplosive subtree contributes $2^{n-|v|} = |J_v|2^n$ vertices to the count M_n, where $|J|$ denotes Lebesgue measure of $J \subset [0,1]$. Notice that non-intersecting hyperexplosive subtrees correspond to the dyadic rationals with non-intersecting interiors. Let

$$\mathscr{H} = \{s \in \partial\mathbb{T} : \text{there is a maximal hyperexploding subtree rooted at } s|m \text{ for some } m \geq 0.\}$$

Then $|\mathscr{H}| = \sum_{v\in\mathbb{T}} |J_v|$ and $|\mathscr{H}| > 0$ since by Proposition 4.8 every exploding tree has a hyperexploding subtree. Let $\epsilon_k = 2^{-k}$, and let $v^1, \ldots, v^{m_k} \in \mathbb{T}$ be a finite set of root vertices of hyperexplosive subtrees, arranged in increasing order of $|v^i|$, such that

$$|\mathscr{H} \setminus \cup_{i=1}^{m_k} J_i| < \epsilon_k, \quad \cup_{i=1}^{m_k} J_{v^i} \subset \mathscr{H}.$$

If $n > m_k$, these hyperexploding subtrees contribute $| \cup_{i=1}^{n} J_{v^i}|2^n$ vertices to the count M_n. Thus,

$$M_n \geq (|\mathscr{H}| - \epsilon_k)2^n.$$

Next consider the t-leaf count. Each t-leaf $v \in \partial V(t)$ corresponds to the dyadic interval $J_v = [x_v, x_v + 2^{|v|}]$, so that distinct t-leaves correspond to intervals having non-overlapping interiors. In this case each J_v contributes *only once* to the count N_n, provided $n \geq |v|$. Let,

$$\mathscr{L} = \cup_{v\in\partial V(t)} J_v.$$

That is, $\mathscr{L}$ is the subset of $[0,1]$ corresponding to the t-leaves. If the tree has at least one t-leaf, and if $S < t$, then $|\mathscr{L}| = \sum_{v\in\partial V} |J_v| > 0$. $\mathscr{L}$ and $\mathscr{H}$ have disjoint interiors as well. For $\epsilon_k = 2^{-k}$, there exist $v^1, \ldots, v^{\ell_k}$, arranged in increasing order of $|v^i|$, such that

$$|\mathscr{L}| - \sum_{i=1}^{\ell_k} |J_{v_i}| < \epsilon_k, \qquad \cup_{i=1}^{\ell_k} J_{v_i} \subset \mathscr{L}.$$

If $n > |v_{\ell_k}|$ then the tree has at least ℓ_k t-leaves, and at most $\ell_k + \epsilon_k 2^n$ intervals of size 2^{-n} on a set of measure at most ϵ_k. Thus,

$$\ell_k \leq N_n \leq \ell_k + \epsilon_k 2^n.$$

Collecting these counts, one has for all $k \geq 1$, there are $m_k, \ell_k \geq 1$ such that for all $n \geq \max\{\ell_k, m_k\}$

$$N_n \leq \ell_k + \epsilon_k 2^n, \qquad M_n \geq (|\mathscr{H}| - \epsilon_k)2^n.$$

Write $u_0 = \delta^{-D}, D > 0$. In the case (i) one has $u_0 = \delta^{-D} > 1$ and $0 < \delta < 1$ so that in the n-th iteration

$$\begin{aligned} X_n(t) &\leq \delta^{-D(\ell_k+\epsilon_k 2^n)} \delta^{(|\mathscr{H}|-\epsilon_k)2^n} \\ &= \delta^{(|\mathscr{H}|-(1+D)\epsilon_k)2^n - \ell_k}, \end{aligned} \tag{5.7}$$

In the case (ii), $u_0 = \delta^{-D} \in [0, 1)$ and $\delta > 1$, so that

$$\begin{aligned} X_n(t) &\geq \delta^{-D(\ell_k+\epsilon_k 2^n)}\delta^{(|\mathscr{H}|-\epsilon_k)2^n} \\ &= \delta^{(|\mathscr{H}|-(1+D)\epsilon_k)2^n-\ell_k}, \end{aligned} \tag{5.8}$$

provided $n \geq \ell_k \wedge m_k$. Note that given the exploding tree, $|\mathscr{H}|, D, \delta$ are fixed positive quantities. So choosing k such that $(|\mathscr{H}|-(1+D))\epsilon_k > 0$, the assertions in the lemma follow in the indicated limits. □

6 Stochastic Picard Ground State Iterations and Stochastic Transforms: the non-uniqueness of α-Riccati solutions

The purpose of this section is to use the stochastic Picard iterations method described below and the non-uniqueness results for (6.1) in Proposition 3.4 to construct a family of non-unique global solutions to (1.2) for any $\alpha > 1$ and a range of initial data u_0. (We note that, as shown in [14], when $\alpha > 1$, solutions of (1.2) blow up in finite time, limiting the range of initial data for which global solutions exist.) Notably, as will be seen in Section 6.2, the α-Riccati equation and the pantograph equation can be connected via a transformation *at the level of solution processes*, which, in the case $\alpha > 2$, allows us to exploit the non-minimal solution process constructed in Theorem 3.5 for the pantograph equation to construct multiple solutions for (1.2). Moreover, in the case of the α-Riccati equation, the non-uniqueness is established for the case $u_0 = 1$, and then transferred to other initial data via a use of another transformation at the level of solution processes, this time connecting solution processes for corresponding to $u_0 = 1$ and arbitrary u_0.

Note that the linearization of (1.2) with $u_0 = 1$, about the constant steady state $u \equiv 1$ is the pantograph equation with $b = -1, a = 2$:

$$v'(t) = -v(t) + 2v(\alpha t), \quad v(0) = 0. \tag{6.1}$$

Our goal is to prove the following theorem.

Theorem 6.1. *Let $\alpha > 1$ and $u_0 \in R_\alpha \subset \mathbb{R}$ defined by*

$$R_\alpha = \big[0, \max\{1, (2\alpha-1)/4 - (6\alpha^2 - 15\alpha + 4)/4(\alpha-1)(2\alpha-1)\}\big) \cup \{1\}. \tag{6.2}$$

For each $\lambda > 0$, there exists a solution u_λ to (1.2) such that

$$\lim_{t\to\infty} \frac{1 - u_\lambda(t)}{t^{-\gamma}} = \lambda, \tag{6.3}$$

where $\gamma = \log_\alpha 2 > 0$.

Consequently, there are infinitely many solutions converging to 1 with an algebraic rate $t^{-\gamma}$ as $t \to \infty$. In the case $\alpha \in (1, 2)$, it was shown in [13] that $u_*(t) = \mathbb{P}(|\partial V(t)| < \infty)$ is also a solution to (1.2) with $u_0 = 1$. This solution has an exponential convergence rate as $t \to \infty$ since

$$1 - u_*(t) = \mathbb{P}(|\partial V(t)| = \infty) \leq \mathbb{P}(L > t) \leq Ce^{-t},$$

where the last inequality is due to Theorem 4.5. Thus, u_* does not belong to the family of solutions $\{u_\lambda\}_{\lambda>0}$.

The proof of Theorem 6.1 will involve several steps. We will first consider the case of $u_0 = 1$ in two separate regimes: $\alpha \in (1, 2]$ and $\alpha > 2$. We then extend the results to other initial data.

6.1 Proof of Theorem 6.1 in the case $u_0 = 1$ and $1 < \alpha \le 2$

A solution u to (1.2) with $u_0 = 1$ is a fixed point of F, where

$$F[u](t) = e^{-t} + \int_0^t e^{-s} u^2(\alpha(t-s))ds.$$

Let

$$\gamma = \log_\alpha 2 = \frac{\ln 2}{\ln \alpha} \ge 1. \tag{6.4}$$

Proposition 6.2. *For sufficiently large $M > M_\alpha > 0$ and sufficiently small $0 < \delta < \delta_M$, the function*

$$\rho_{M,\delta}(t) = \begin{cases} 1 & \text{if} \quad t \le M, \\ 1 - \delta t^{-\gamma} & \text{if} \quad t > M \end{cases} \tag{6.5}$$

satisfies $F[\rho_{M,\delta}] \le \rho_{M,\delta}$.

Proof. To simplify the notation in this proof, we will drop the subscripts of $\rho_{M,\delta}$. Let

$$\tilde\rho(t) = 1 - \rho(t) = \begin{cases} 0 & \text{if} \quad t \le M, \\ \delta t^{-\gamma} & \text{if} \quad t > M \end{cases}$$

and

$$G[\tilde\rho] = 1 - F[\tilde\rho] = \int_0^t e^{-(t-s)}(2\tilde\rho(\alpha s) - \tilde\rho(\alpha s)^2)ds.$$

It suffices to show that $G[\tilde\rho] \ge \tilde\rho$. For $t \ge M$,

$$\begin{aligned} G[\tilde\rho] = \int_0^t e^{-t+s}(2\tilde\rho(\alpha s) - \tilde\rho(\alpha s)^2)ds &= \int_{M/\alpha}^t e^{-t+s}(2\delta\alpha^{-\gamma}s^{-\gamma} - \delta^2\alpha^{-2\gamma}s^{-2\gamma})ds \\ &= 2\delta\alpha^{-\gamma}e^{-t}\int_{M/\alpha}^t e^s s^{-\gamma}\left(1 - \frac{\delta\alpha^{-\gamma}}{2}s^{-\gamma}\right)ds \\ &= \delta e^{-t}\int_{M/\alpha}^t e^s s^{-\gamma}\left(1 - \frac{\delta}{4}s^{-\gamma}\right)ds. \end{aligned}$$

The inequality $G[\tilde\rho] \ge \tilde\rho$ will be held for all $t \ge 0$ provided that

$$\int_{M/\alpha}^t e^s s^{-\gamma}\left(1 - \frac{\delta}{4}s^{-\gamma}\right)ds \ge e^t t^{-\gamma} \qquad \forall t \ge M.$$

This inequality is equivalent to

$$\frac{\delta}{4} \le f(t) := \frac{\int_{M/\alpha}^t e^s s^{-\gamma}ds - e^t t^{-\gamma}}{\int_{M/\alpha}^t e^s s^{-2\gamma}ds} \qquad \forall t \ge M. \tag{6.6}$$

We will show that for sufficiently large M, $\inf_{t\ge M} f(t) > 0$. Once this is proven, (6.6) will be satisfied by choosing $\delta \le \delta_M = 4\inf_{t\ge M} f(t)$. By L'Hospital Rule,

$$\lim_{t\to\infty} f(t) = \lim_{t\to\infty} \frac{\gamma e^t t^{-\gamma-1}}{e^t t^{-2\gamma}} = \begin{cases} 1 & \text{if} \quad \gamma = 1 \\ \infty & \text{if} \quad \gamma > 1. \end{cases}$$

Since f is continuous on $(0, \infty)$, showing $\inf_{t \geq M} f(t) > 0$ is equivalent to showing that $f(t) > 0$ for all $t \geq M$. Note that $f(t)$ has the same sign as

$$f_1(t) = \int_{M/\alpha}^{t} e^s s^{-\gamma} ds - e^t t^{-\gamma}.$$

Since $f_1'(t) = \gamma e^t t^{-\gamma-1} > 0$, one sees that $f_1(t) \geq f_1(M)$ for all $t \geq M$. On the other hand, $f_1(M) = f_2(M)$ where

$$f_2(x) = \int_{x/\alpha}^{x} e^s s^{-\gamma} ds - e^x x^{-\gamma}.$$

One has $f_2'(x) = \gamma e^x x^{-\gamma-1} - \frac{1}{\alpha} e^{x/\alpha}(x/\alpha)^{-\gamma}$. Note that $f_2'(x) > 1$ for sufficiently large x and hence, $\lim_{x\to\infty} f_2(x) = \infty$. Hence, there exists $M_\alpha > 0$ such that $f_2(M) > 0$ for all $M > M_\alpha$. Therefore, $f(t) > 0$ for all $t \geq M > M_\alpha$. ☐

Proposition 6.3. *Let $M, \delta > 0$ and $\rho = \rho_{M,\delta}$ be defined as in Proposition 6.2. Consider stochastic Picard iterations $X_n(t) = X_{M,\delta,n}(t)$ with the ground state $X_0(t) = X_{M,\delta,0}(t) = \rho_{M,\delta}(t)$ and the initial state $u_0 = 1$, i.e.*

$$X_0(t) = \rho(t), \quad X_n(t) = \begin{cases} 1 & \text{if } \; T_\theta \geq t, \\ X_{n-1}^{(1)}(\alpha(t - T_\theta))\, X_{n-1}^{(2)}(\alpha(t - T_\theta)) & \text{if } \; T_\theta < t. \end{cases}, n \geq 1, \tag{6.7}$$

where $X_{n-1}^{(1)}$ and $X_{n-1}^{(2)}$ are conditionally on T_θ i.i.d. copies of X_{n-1}. Then :

(i) For all $n \in \mathbb{N}$ and $t \geq 0$

$$X_n(t) = \prod_{v \in \mathbb{T},\ |v|=n-1} \rho^2_{M,\delta}(\alpha^n(t - \Theta_v)), \qquad a.s. \tag{6.8}$$

where $\Theta_v = \sum_{j=0}^{|v|} \frac{T_{v|j}}{\alpha^j}$.

(ii) For each $t > 0$, The sequence $\{X_n(t)\} = \{X_{M,\delta,n}(t)\}$ is a non-negative supermartingale with respect to the filtration $\mathscr{F}_n = \sigma(T_v : \ |v| \leq n-1)$.

Proof. The formula (6.8) from part *(i)* follows by induction from the stochastic iterations. Indeed, for $n = 1$, since $\rho(t) = 1$ for $t \leq 0$ we have

$$X_1(t) \overset{(6.7)}{=} \mathbb{1}_{[T_\theta \geq t]} + \rho^2(\alpha(t - T_\theta))\mathbb{1}_{[T_\theta < t]} = \rho^2(\alpha(t - T_\theta)),$$

So (6.8) holds. The inductive step follows similarly, once we observe that the product in the right-hand side of (6.8) is 1 if $t \leq 0$,

To prove *(ii)*, we will show by induction on n that $\mathbb{E}[X_{n+1}(t)|\mathscr{F}_n] \leq X_n(t)$. For $n = 1$, $\mathscr{F}_1 = \sigma(T_\theta)$ and, as noted above, $X_1(t) = \rho^2(\alpha(t - T_\theta))$. Note that $\mathscr{F}_0 = \emptyset$. Thus, since $\rho(t) = 1$ for $t \leq 0$, using Proposition 6.2, we obtain

$$\begin{aligned} \mathbb{E}(X_1(t)|\mathscr{F}_0) = \mathbb{E}(X_1(t)) &= \int_0^\infty e^{-s}\rho^2(\alpha(t-s))\, ds \\ &= e^{-t} + \int_0^t e^{-s}\rho^2(\alpha(t-s))\, ds \\ &\leq \rho(t) = X_0(t) \quad \text{for all } t \geq 0. \end{aligned} \tag{6.9}$$

For $n = 2$, using (6.8), we write

$$X_2(t) = \prod_{|v|=1} \rho^2(\alpha^2(t - \Theta_v)) = \prod_{|v|=1} \rho^2(\alpha(\tau - T_v)) = X_1^{(1)}(\tau) X_1^{(2)}(\tau)$$

where $\tau = \alpha(t - T_\theta)$ and

$$X_1^{(1)}(\tau) = \mathbb{1}_{[T_1 \geq \tau]} + \rho^2(\alpha(\tau - T_1))\mathbb{1}_{[T_1 < \tau]} = \rho^2(\alpha(\tau - T_1)),$$
$$X_1^{(2)}(\tau) = \mathbb{1}_{[T_2 \geq \tau]} + \rho^2(\alpha(\tau - T_2))\mathbb{1}_{[T_2 < \tau]} = \rho^2(\alpha(\tau - T_2)).$$

Note that, conditionally on T_θ, $X_1^{(1)}(\tau)$ and $X_1^{(2)}(\tau)$ are i.i.d. and have the same distribution as $X_1(\tau)$. Therefore, using the substitution property for conditional expectations together with (6.9), we have:

$$\begin{aligned}\mathbb{E}(X_2(t)|T_\theta) &= \mathbb{E}(X_1^{(1)}(\tau)X_1^{(2)}(\tau)|\tau) = \mathbb{E}(X_1^{(1)}(\tau)|\tau) \cdot \mathbb{E}(X_1^{(2)}(\tau)|\tau) \\ &= \mathbb{E}(X_1(\tau)|\tau)^2 \leq \rho^2(\tau) = \rho^2(\alpha(t - T_\theta)) = X_1(t). \end{aligned} \tag{6.10}$$

Now suppose $\mathbb{E}(X_n(t)|\mathscr{F}_{n-1}) \leq X_{n-1}(t)$ for some $n \geq 2$. Using (6.7) together with the fact that $X_n(t) = 1$ for $t \leq 0$ (as follows from (6.8)), we have

$$X_{n+1}(t) = X_n^{(1)}(\tau)\, X_n^{(2)}(\tau),$$

where, as before, $\tau = \alpha(t - T_\theta)$. Recall that, conditionally on T_θ, $X_n^{(1)}(\tau)$ and $X_n^{(2)}(\tau)$ are independent and distributed as $X_n(\tau)$. For $k \in \{1, 2\}$, denote $\mathscr{F}_{n-1}^{(k)} = \sigma(T_{kv} : |v| = n - 2)$. Because $\mathscr{F}_n = \sigma(T_\theta) \vee \mathscr{F}_{n-1}^{(1)} \vee \mathscr{F}_{n-1}^{(2)}$ and that $\sigma(T_\theta)$, $\mathscr{F}_{n-1}^{(1)}$ and $\mathscr{F}_{n-1}^{(2)}$ are independent, we get by organizing random variables according to

$$(T_v : |v| \leq n - 1) = (\, T_\theta, (\, (T_{1v} : |v| \leq n - 2), (T_{2v} : |v| \leq n - 2)\,)\,),$$

and applying the substitution property in two directions, as in (6.10), followed by the induction hypothesis,

$$\begin{aligned} &\mathbb{E}(X_{n+1}(t)|\mathscr{F}_n) \\ = \;& \mathbb{1}_{[T_\theta \geq t]} + \mathbb{E}(X_n^{(1)}(\tau)X_n^{(2)}(\tau)\mathbb{1}_{[T_\theta < t]}|\mathscr{F}_n) \\ = \;& \mathbb{1}_{[T_\theta \geq t]} + \mathbb{E}(\mathbb{1}_{[T_\theta < t]}X_n^{(1)}(\tau)|\sigma(T_\theta) \vee \mathscr{F}_{n-1}^{(1)}) \cdot \mathbb{E}(\mathbb{1}_{[T_\theta < t]}X_n^{(2)}(\tau)|\sigma(T_\theta) \vee \mathscr{F}_{n-1}^{(2)}) \\ \leq \;& \mathbb{1}_{[T_\theta \geq t]} + X_{n-1}^{(1)}(\tau)X_{n-1}^{(2)}(\tau)\mathbb{1}_{[T_\theta < t]} = X_n(t). \end{aligned} \tag{6.11}$$

□

Proposition 6.4. *Assume $\alpha \in (1, 2]$. Let $M, \delta > 0$ and $\rho_{M,\delta}$ be defined as in Proposition 6.2 and $\{X_{M,\delta,n}(t)\}_{n\geq 1}$ be the stochastic process defined by (6.8).*

(i) As $n \to \infty$, $X_{M,\delta,n}(t)$ converges a.s. and the limit process $X_{M,\delta}(t) = \lim_{n\to\infty} X_{M,\delta,n}(t)$ satisfies (2.16) with $u_0 = 1$. Moreover, $0 < X_{M,\delta}(t) < 1$ on the event $[L < t]$, where L denotes the longest path, see (4.6).

(ii) The function $u_{M,\delta}(t) = \mathbb{E}(X_{M,\delta}(t))$ *solves* (1.2) *with* $u_0 = 1$. *Moreover,*

$$\lim_{t\to\infty} \frac{1-u_{M,\delta}(t)}{t^{-\gamma}} \quad = \quad \delta\,.$$

Proof. To simplify the notations in the proof, we will drop the subscripts M and δ and will only keep the subscript n.

By Proposition 6.3 (ii) and Doob's Martingale Convergence Theorem for positive supermartingales, the sequence $\{X_n(t)\} = \{X_{M,\delta,n}(t)\}_{n\geq 1}$ converges a.s.. Denote the limit by $X(t) = X_{M,\delta}(t)$. Since for all $n \in \mathbb{N}$ and $t > 0$, $X_n(t) \in [0,1]$ a.s., we have $\mathbb{E}(X(t)) \in [0,1]$. By Theorem 5.1, $X(t)$ satisfies (2.16) with $u_0 = 1$. Thus $u(t) = \mathbb{E}(X(t))$ satisfies (1.2).

On the event $[L < t]$, one has

$$X_n(t) = \prod_{|v|=n-1} \rho^2(\alpha^n(t-\Theta_v)), \quad \forall\, n \geq 1.$$

Since $t-\Theta_v \geq t-L > 0$, we have $\alpha^n(t-\Theta_v) > M$ for sufficiently large n independent on $v \in \mathbb{T}$. Here, M is the number in Proposition 6.2. Thus, for sufficiently large n,

$$X_n(t)\mathbb{1}_{[L<t]} = \prod_{|v|=n-1} (1-\delta(\alpha^n(t-\Theta_v))^{-\gamma})^2 \mathbb{1}_{[L<t]} = \prod_{|v|=n-1} \left(1-\delta\frac{(t-\Theta_v)^{-\gamma}}{2^n}\right)^2 \mathbb{1}_{[L<t]}$$

Denote $S_n = \min_{|v|=n} \Theta_v$ and $L_n = \max_{|v|=n} \Theta_v$. Note that $S_{n-1} \leq \Theta_v \leq L_{n-1}$ for $|v| = n-1$. Thus,

$$\prod_{|v|=n-1} \left(1-\delta\frac{(t-L_n)^{-\gamma}}{2^n}\right)^2 \mathbb{1}_{[L<t]} \leq X_n(t)\mathbb{1}_{[L<t]} \leq \prod_{|v|=n-1} \left(1-\delta\frac{(t-S_n)^{-\gamma}}{2^n}\right)^2 \mathbb{1}_{[L<t]}$$

In other words,

$$\left(1-\delta\frac{(t-L_n)^{-\gamma}}{2^n}\right)^{2\cdot 2^{n-1}} \mathbb{1}_{[L<t]} \leq X_n(t)\mathbb{1}_{[L<t]} \leq \left(1-\delta\frac{(t-S_n)^{-\gamma}}{2\cdot 2^n}\right)^{2^{n-1}} \mathbb{1}_{[L<t]}.$$

Letting $n \to \infty$, one gets

$$e^{-\delta(t-L)^{-\gamma}}\mathbb{1}_{[L<t]} \leq X(t)\mathbb{1}_{[L<t]} \leq e^{-\delta(t-S)^{-\gamma}}\mathbb{1}_{[L<t]}\,. \tag{6.12}$$

Therefore, $0 < X(t) < 1$ on the event $[L < t]$.

To establish the limit in part *(ii)*, we estimate

$$\mathbb{E}(X(t)\mathbb{1}_{[L<t]}) \leq \mathbb{E}(X(t)) = \mathbb{E}(X(t)\mathbb{1}_{[L<t]}) + \mathbb{E}(X(t)\mathbb{1}_{[L\geq t]}) \leq \mathbb{E}(X(t)\mathbb{1}_{[L<t]}) + \mathbb{E}(\mathbb{1}_{[L\geq t]}).$$

Together with (6.12), we have

$$\mathbb{E}\left(e^{-\delta(t-L)^{-\gamma}}\mathbb{1}_{[L<\epsilon t]}\right) \leq u(t) \leq \mathbb{E}\left(e^{-\delta(t-S)^{-\gamma}}\mathbb{1}_{[L<t]}\right) + \mathbb{E}(\mathbb{1}_{[L\geq t]})$$

for any constant $\epsilon \in (0,1)$. Hence,

$$\mathbb{E}\left((1-e^{-\delta(t-S)^{-\gamma}})\mathbb{1}_{[L<t]}\right) \leq 1-u(t) \leq \mathbb{E}\left((1-e^{-\delta(t-L)^{-\gamma}})\mathbb{1}_{[L<\epsilon t]}\right) + \mathbb{E}(\mathbb{1}_{[L\geq \epsilon t]}).$$

By Theorem 4.5, one has $\mathbb{E}(\mathbb{1}_{[L\geq\epsilon t]}) \leq Ce^{-\epsilon t}$ for all $t > 0$. Dividing both sides of the above inequalities by $t^{-\gamma}$, we have

$$\mathbb{E}\left(\frac{1-e^{-\delta(t-S)^{-\gamma}}}{t^{-\gamma}}\mathbb{1}_{[L<t]}\right) \leq \frac{1-u(t)}{t^{-\gamma}} \leq \mathbb{E}\left(\frac{1-e^{-\delta(t-L)^{-\gamma}}}{t^{-\gamma}}\mathbb{1}_{[L<\epsilon t]}\right) + Ct^{\gamma}e^{-\epsilon t} \tag{6.13}$$

Note that almost surely

$$\lim_{t\to\infty} \frac{1-e^{-\delta(t-S)^{-\gamma}}}{t^{-\gamma}} = \delta$$

Also, $\lim_{t\to\infty} \mathbb{1}_{[L<t]} = \mathbb{1}_{[L<\infty]} = 1$ a.s. due to hyperexplosion [14]. By Fatou's Lemma,

$$\liminf_{t\to\infty} \frac{1-u(t)}{t^{-\gamma}} \geq \mathbb{E}\left(\lim_{t\to\infty} \frac{1-e^{-\delta(t-S)^{-\gamma}}}{t^{-\gamma}}\mathbb{1}_{[L<t]}\right) = \delta.$$

By the inequality $1-e^{-x} \leq x$, one has

$$\begin{aligned} \text{RHS}(6.13) \leq \mathbb{E}\left(\frac{\delta(t-L)^{-\gamma}}{t^{-\gamma}}\mathbb{1}_{L<\epsilon t}\right) + Ct^{\gamma}e^{-\epsilon t} \quad &\leq \quad \mathbb{E}\left(\frac{\delta(t-\epsilon t)^{-\gamma}}{t^{-\gamma}}\mathbb{1}_{L<\epsilon t}\right) + Ct^{\gamma}e^{-\epsilon t} \\ &\leq \quad \delta(1-\epsilon)^{-\gamma} + Ct^{\gamma}e^{-\epsilon t} \end{aligned}$$

Thus,

$$\limsup_{t\to\infty} \frac{1-u(t)}{t^{-\gamma}} \leq \delta(1-\epsilon)^{-\gamma}$$

Because this inequality is true for all $\epsilon \in (0,1)$, one has

$$\limsup_{t\to\infty} \frac{1-u(t)}{t^{-\gamma}} \leq \delta,$$

which completes the proof. □

Theorem 6.5. *Let $M, \delta > 0$ and $\rho_{M,\delta}$ be defined as in Proposition 6.2, and $X_{M,\delta}(t)$ be the process defined in Proposition 6.4. Then, for any $\lambda \geq 0$, the process $X_{M,\delta,\lambda}(t) = (X_{M,\delta}(t))^{\lambda/\delta}$ is a solution process satisfying (2.16) with $u_0 = 1$, and the function $u_{M,\delta,\lambda}(t) = \mathbb{E}(X_{M,\delta,\lambda}(t))$ solves the problem (1.2) with $u_0 = 1$. Moreover:*

(i)

$$\lim_{t\to\infty} \frac{1-u_{M,\delta,\lambda}(t)}{t^{-\gamma}} = \lambda. \tag{6.14}$$

(ii) For any $t > 0$,

$$\begin{aligned} u_{M,\delta,\lambda}(t) \quad &> \quad u_{M,\delta,\lambda'}(t) \quad \text{if } 0 \leq \lambda < \lambda', \\ u_{M,\delta,\lambda}(t) \quad &\leq \quad u_{M',\delta,\lambda}(t) \quad \text{if } M < M',\ 0 < \delta < \min\{\delta_M, \delta_{M'}\} \\ u_{M,\delta,\lambda}(t) \quad &\geq \quad u_{M,\delta',\lambda}(t) \quad \text{if } 0 < \delta < \delta' < \delta_M. \end{aligned}$$

Proof. The fact that $X_{M,\delta,\lambda}(t)$ is a solution process satisfying (2.16) with $u_0 = 1$, follows from raising to power α/δ both sides of (2.16) with $X = X_{M,\delta}$ and $u_0 = 1$. Thus, $u_{M,\delta,\lambda} = \mathbb{E}(X_{M,\delta,\lambda}(t))$ satisfies (1.2) with $u_0 = 1$, since, as it will be shown below, the expectation is finite.

To prove *(i)*, raise the equation (6.12) to power λ/δ to obtain

$$e^{-\lambda(t-L)^{-\gamma}}\mathbb{1}_{[L<t]} \le (X_{M\delta}(t))^{\lambda/\delta}\mathbb{1}_{[L<t]} \le e^{-\lambda(t-S)^{-\gamma}}\mathbb{1}_{[L<t]}\,. \tag{6.15}$$

From here, one can follow the same lines of the proof of Proposition 6.4, part *(ii)*, to show that $u_{M,\delta,\lambda}(t)$ is finite and

$$\lim_{t\to\infty}\frac{1-u_{M,\delta,\lambda}(t)}{t^{-\gamma}} = \lambda.$$

To prove *(ii)*, suppose $0 \le \lambda < \lambda'$. Because $0 < X_{M,\delta}(t) < 1$ on the event $[L < t]$, $X^{\lambda'/\delta}_{M,\delta}(t) < X^{\lambda/\delta}_{M,\delta}(t)$ on this event. Since $[L < t]$ is not a null event for any $t > 0$, one has $u_{M,\delta,\lambda'}(t) < u_{M,\delta,\lambda}(t)$.

Next, suppose $M < M'$ and $0 < \delta < \min\{\delta_M, \delta_{M'}\}$. From the definition of $\rho_{M,\delta}$ in Proposition 6.2, it is clear that $\rho_{M,\delta}(t) \le \rho_{M',\delta}(t)$, which leads to $X_{M,\delta,n}(t) \le X_{M',\delta,n}(t)$ for all n. Therefore, $u_{M,\delta,\lambda}(t) \le u_{M',\delta,\lambda}(t)$.

Next, suppose $0 < \delta < \delta' < \delta_M$. Denote $\kappa = \delta'/\delta > 1$. Note that for $t > M$,

$$\rho^{\kappa}_{M,\delta}(t) = (1-\delta t^{-\gamma})^{\kappa} > 1-\kappa\delta t^{-\gamma} = \rho_{M,\delta'}(t).$$

Thus, $X^{\kappa}_{M,\delta,n}(t) \ge X_{M,\delta',n}(t)$ for all n. Raising both sides to power λ/δ', one gets $X^{\lambda/\delta}_{M,\delta,n}(t) \ge X^{\lambda/\delta'}_{M,\delta',n}(t)$. Hence, $u_{M,\delta,\lambda}(t) \ge u_{M,\delta',\lambda}(t)$. □

6.2 Proof of Theorem 6.1 in the case $u_0 = 1$ and $\alpha > 2$

This involves two basic ideas: (i) A transform of the solution process for the multiplicative α-Riccati model, and (ii) a stochastic Picard iterations with special ground state X_0. All such solutions have an exact convergence rate $1 - u(t) \sim t^{-\gamma}$ as $t \to \infty$, where, as in (6.4) or in (3.4) with $a = 2$,

$$\gamma = \gamma(\alpha, a = 2) = \log_\alpha 2 = \frac{\ln 2}{\ln \alpha} \in (0, 1). \tag{6.16}$$

As a consequence, $1 - u \notin L^1$.

Proposition 6.6. *Suppose $\mathcal{X}(t) \ge 0$ is a nonnegative binary solution process for (6.1), i.e.*

$$\mathcal{X}(t) = \begin{cases} 0 & \text{if } \; T_\theta \ge t \\ \mathcal{X}^{(1)}(\alpha(t-T_\theta)) + \mathcal{X}^{(2)}(\alpha(t-T_\theta)) & \text{if } \; T_\theta < t \end{cases} \tag{6.17}$$

where $T \sim Exp(1)$ and $\mathcal{X}^{(1)}$, $\mathcal{X}^{(2)}$ are two, conditionally on T_θ, i.i.d. copies of $\mathcal{X}(t)$ (see also (2.17) with $a = 2$, $u_0 = 0$). Then

(i) $v(t) = \mathbb{E}(\mathcal{X}(t))$, *if finite for all t, satisfies (6.1).*

(ii) *For any $\lambda \ge 0$, $X(t) = e^{-\lambda\mathcal{X}(t)}$ satisfies (2.16) and $u(t) = \mathbb{E}\left(e^{-\lambda\mathcal{X}(t)}\right)$ satisfies (1.2).*

Proof. By conditioning on T_θ in (6.17) we get

$$v(t) = \int_0^t e^{-s} 2v(\alpha(t-s))ds$$

which leads to (6.1), proving *(i)*.

To prove *(ii)*, first apply exponential $e^{-\lambda \cdot}$ to both sides of (6.17) to show $X(t) = e^{-\lambda \mathcal{X}(t)}$ satisfies (2.16). Note that u is always well-defined because $e^{-\lambda \mathcal{X}(t)} \in [0,1]$. By conditioning on T_θ in (2.16), we get

$$u(t) = e^{-t} + \int_0^t e^{-s} u^2(\alpha(t-s))ds$$

which leads to (1.2). ☐

One can observe from Proposition 6.6 that if (6.17) has a solution $\mathcal{X}(t) \geq 0$, not identically zero, then $u_\lambda(t) = \mathbb{E}\left(e^{-\lambda \mathcal{X}(t)}\right)$, $\lambda \geq 0$, is an infinite family of solutions to (1.2) corresponding to $u_0 = 1$. Thus, our next goal is will construct a solution process $\mathcal{X}(t) \geq 0$ of (6.17) that is not identically zero. The key idea is to use the expected value of the unary solution process given by Theorem 3.5 as the ground state in the stochastic Picard iterations for (6.17).

Proposition 6.7. *Let $\eta(t)$ be from Theorem 3.5 with the convention that $\eta(t) = 0$ if $t \leq 0$. On the full binary tree $\mathbb{T}$, define*

$$\mathcal{X}_n(t) = \sum_{|v|=n-1} 2\eta(\alpha^n(t - \Theta_v)), \quad \forall\, n \geq 1 \tag{6.18}$$

where $\Theta_v = \sum_{j=0}^{|v|} \frac{T_{v|j}}{\alpha^j}$. Then

(i) The sequence $\{\mathcal{X}_n(t)\}$ satisfies the stochastic Picard iterations (5.3) for the binary pantograph process with ground state $\mathcal{X}_0(t) = \eta(t)$, corresponding to $a = 2$ and $u_0 = 0$, i.e.

$$\mathcal{X}_n(t) = \begin{cases} 0 & \text{if } \; T_\theta \geq t, \\ \mathcal{X}^{(1)}_{n-1}(\alpha(t-T_\theta)) + \mathcal{X}^{(2)}_{n-1}(\alpha(t-T_\theta)) & \text{if } \; T_\theta < t. \end{cases} \tag{6.19}$$

(ii) For each $t > 0$, $\{\mathcal{X}_n(t)\}$ is a martingale with respect to the filtration $\mathscr{F}_n = \sigma(T_v : |v| \leq n)$.

(iii) The limit $\mathcal{X}(t) = \lim_{n\to\infty} \mathcal{X}_n(t)$ exists. Moreover, $\mathbb{E}(\mathcal{X}(t)) = \eta(t)$. Moreover, for any $\delta \in (1, 1/\gamma)$, $\mathbb{E}(\mathcal{X}^\delta(t)) \leq \frac{\eta_\delta(t)}{2^\delta}$, where η_δ is given in Remark 3.7.

Proof. Note that if the ground state $X_0(t) = 0$ for $t \leq 0$, the iterative formula (6.19) can be re-written as

$$\mathcal{X}_n(t) = \mathcal{X}^{(1)}_{n-1}(\alpha(t-T_\theta)) + \mathcal{X}^{(1)}_{n-1}(\alpha(t-T_\theta)), \tag{6.20}$$

Since by induction $\mathcal{X}_n(t) = 0$ on $t \in (-\infty, 0]$ for all n. In case $\mathcal{X}_0(t) = \eta(t)$, the formula for $\mathcal{X}_n$ given by (6.18) follows from (6.20) directly from the definition of $\mathcal{X}_n(t)$ also by induction. Thus, (6.18) satisfies (6.19).

Next, we will show by induction on n that $\mathbb{E}(\mathcal{X}_{n+1}(t)|\mathscr{F}_n) = \mathcal{X}_n(t)$ following the same approach as in the proof of Proposition 6.3, part *(ii)*. Namely, for $n = 1$, $\mathscr{F}_0 = \sigma(\emptyset)$ and $\mathcal{X}_1(t) = 2\eta(\alpha(t - T_\theta))$, so

$$\mathbb{E}(\mathcal{X}_1 \,|\, \mathscr{F}_0) = \mathbb{E}(\mathcal{X}_1) = \int_0^t e^{-s} 2\eta(\alpha(t-s))\, ds = \eta(t) = X_0(t). \tag{6.21}$$

since by Theorem 3.5, η satisfies (6.1).

In the case $n = 2$, $\mathscr{F}_1 = \sigma(T_\theta)$, and thus

$$\begin{aligned}\mathbb{E}(\mathcal{X}_2 \,|\, \mathscr{F}_1) &= \mathbb{E}(2\eta(\alpha^2(t-\Theta_1)) + 2\eta(\alpha^2(t-\Theta_2)) \,|\, T_\theta) \\ &= \mathbb{E}(2\eta(\alpha(\tau - T_1)) + 2\eta(\alpha(\tau - T_2)) \,|\, T_\theta)\end{aligned}$$

where $\tau = \alpha(t - T_\theta)$. Thus,

$$\begin{aligned}\mathbb{E}(\mathcal{X}_2|\mathscr{F}_1) &= \mathbb{E}(2\eta(\alpha(\tau - T_1) + 2\eta(\alpha(\tau - T_2)) \,|\, T_\theta) \\ &= \mathbb{E}(\mathcal{X}_1^{(1)}(\tau) \,|\, \tau) + \mathbb{E}(\mathcal{X}_1^{(2)}(\tau) \,|\, \tau) = 2\,\mathbb{E}(\mathcal{X}_1(\tau) \,|\, \tau) = 2sX_0(\tau) = \mathcal{X}_1(t),\end{aligned}$$

where (6.21) and the substitution property for conditional probability was used in the 2nd to the last equality.

Now suppose $\mathbb{E}(\mathcal{X}_n|\mathscr{F}_{n-1}) = \mathcal{X}_{n-1}$ for some $n \geq 2$. We have

$$\mathbb{E}(\mathcal{X}_{n+1}|\mathscr{F}_n) = \mathbb{E}(\mathcal{X}_n^{(1)}(\alpha(t - T_\theta)) + \mathcal{X}_n^{(2)}(\alpha(t - T_\theta)) \,|\, \mathscr{F}_n).$$

Because $\mathscr{F}_n = \sigma(T_0) \vee \mathscr{F}_{n-1}^{(1)} \vee \mathscr{F}_{n-1}^{(2)}$ and that $\mathscr{F}_{n-1}^{(1)}$ and $\mathscr{F}_{n-1}^{(2)}$ are independent, using substitution property, we get

$$\begin{aligned}\mathbb{E}(\mathcal{X}_{n+1}|\mathscr{F}_n) &= \mathbb{E}(\mathcal{X}_n^{(1)}(\alpha(t - T_\theta)) \,|\, T_\theta, F_{n-1}^{(1)}) + \mathbb{E}(\mathcal{X}_n^{(2)}(\alpha(t - T_\theta)) \,|\, T_\theta, \mathscr{F}_{n-1}^{(2)}) \\ &= \mathcal{X}_{n-1}^{(1)}(\alpha(t - T_\theta)) + \mathcal{X}_{n-1}^{(2)}(\alpha(t - T_\theta)) = \mathcal{X}_n(t)\,,\end{aligned}$$

which proves *(ii)*.

To prove *(iii)*, we use the fact that $\mathcal{X}_n(t) \geq 0$ and $\{\mathcal{X}_n(t)\}_{n\geq 1}$ is a martingale, which implies that for any $t > 0$, $\mathcal{X}_n(t)$ is convergent a.s. to some process $\mathcal{X}(t)$. By Theorem 5.1, $\mathcal{X}$ is a solution process satisfying

$$\mathcal{X}(t) = \begin{cases} 0 & \text{if } \; T_\theta \geq t, \\ \mathcal{X}^{(1)}(\alpha(t - T_\theta)) + \mathcal{X}^{(2)}(\alpha(t - T_\theta)) & \text{if } \; T_\theta < t. \end{cases}$$

Also,

$$\mathbb{E}(\mathcal{X}_1(t)) = \mathbb{E}(2\eta(\alpha(t - T_\theta))) = \int_0^t 2\eta(\alpha(t-s))e^{-s} ds = \eta(t).$$

By the martingale property, $\mathbb{E}(\mathcal{X}_n(t)) = \eta(t)$ for all $n \geq 1$. To show that $\mathbb{E}(\mathcal{X}(t)) = \eta(t)$, it suffices to show that for each $t > 0$, the sequence $\mathbb{E}(\mathcal{X}_n(t)^\delta)$ is bounded from above for some $\delta > 1$.

Fix $\delta \in (1, 1/\gamma)$. Let η_δ be the function from Remark 3.7, i.e. $\eta_\delta(t) = \mathbb{E}(\tilde{\mathcal{X}}_*^\delta(t))$ with $\tilde{\mathcal{X}}_* = (t - \tilde{S})^{-\gamma} \mathbb{1}_{[\tilde{S} < t]}$ (recall, $\eta(t) = \mathbb{E}(\tilde{\mathcal{X}}_*(t))$). By Jensen's inequality,

$$\eta_\delta(t) \geq \left(\mathbb{E}(\tilde{\mathcal{X}}_*)\right)^\delta = \eta(t)^\delta.$$

We will show by induction on $n \geq 1$ that $\mathbb{E}(\mathcal{X}_n(t)^\delta) \leq \frac{\eta_\delta(t)}{2^\delta}$. For $n = 1$,

$$\begin{aligned}\mathbb{E}(\mathcal{X}_1(t)^\delta) &= \mathbb{E}(\eta(\alpha(t - T_\theta))^\delta) \leq \mathbb{E}[\eta_\delta(\alpha(t - T_\theta))] \\ &= \int_0^t e^{-s} \eta_\delta(\alpha(t-s)) ds = \frac{\eta_\delta(t)}{2^\delta}\end{aligned}$$

according to Remark 3.7. Suppose $\mathbb{E}(\mathcal{X}_{n-1}(t)^\delta) \leq \frac{\eta_\delta(t)}{2^\delta}$ for some $n \geq 2$. Using the inequality $(a+b)^\delta \leq 2^{\delta-1}(a^\delta + b^\delta)$, we have

$$\mathcal{X}_n(t)^\delta \leq \begin{cases} 0 & \text{if } \ T_\theta \geq t, \\ 2^{\delta-1}\left(\mathcal{X}_{n-1}^{(1)}(\alpha(t-T_\theta))^\delta + \mathcal{X}_{n-1}^{(2)}(\alpha(t-T_\theta))^\delta\right) & \text{if } \ T_\theta < t. \end{cases}$$

Thus,

$$\begin{aligned}\mathbb{E}(\mathcal{X}_n(t)^\delta) &\leq 2^{\delta-1} \int_0^t e^{-s} 2\, \mathbb{E}(\mathcal{X}_{n-1}(\alpha(t-s))^\delta) ds = 2^{\delta-1} \int_0^t e^{-s} 2 \frac{\eta_\delta(\alpha(t-s))}{2^\delta} ds \\ &= \int_0^t e^{-s} \eta_\delta(\alpha(t-s)) ds = \frac{\eta_\delta(t)}{2^\delta}. \end{aligned} \tag{6.22}$$

Thus, the sequence $\mathcal{X}_n$ is uniformly integrable and so $\mathbb{E}(\mathcal{X}(t)) = \lim_{n\to\infty} \mathbb{E}(\mathcal{X}_n(t)) = \eta(t)$, while the inequality for $\mathbb{E}(\mathcal{X}(t)^\delta)$ from part *(iii)* follows from (6.22) by taking $n \to \infty$ and applying Fatou's Lemma. □

Theorem 6.8. *Let $\mathcal{X}(t)$ be given by Proposition 6.7, part* (iii). *For each $\lambda \geq 0$, the process $X_\lambda(t) = e^{-\lambda \mathcal{X}(t)}$ is a solution process satisfying* (2.16) *with $u_0 = 1$, while the function $u_\lambda(t) = \mathbb{E}(X_\lambda(t))$ solves* (1.2). *Moreover, for $\lambda > 0$,*

$$\lim_{t\to\infty} \frac{1 - u_\lambda(t)}{t^{-\gamma}} = \lambda.$$

Proof. The fact that with $u_0 = 1$, $X_\lambda(t)$ satisfies (2.16) and $u_\lambda(t)$ solves (1.2) comes directly from Proposition 6.6. We only need to show the convergence rate. On one hand,

$$1 - u_\lambda(t) = \mathbb{E}(1 - e^{-\lambda \mathcal{X}(t)}) \leq \mathbb{E}(\lambda \mathcal{X}(t)) = \lambda \mathbb{E}(\mathcal{X}(t)) = \lambda \eta(t).$$

Then, by Theorem 3.5,

$$\limsup_{t\to\infty} \frac{1 - u_\lambda(t)}{t^{-\gamma}} = \limsup_{t\to\infty} \frac{\eta(t)}{t^{-\gamma}} = \lambda.$$

It remains to show that

$$\liminf_{t\to\infty} \frac{1 - u_\lambda(t)}{t^{-\gamma}} \geq \lambda.$$

Since $\gamma \in (0,1)$, there exists $\delta \in (1,2)$ such that $\gamma\delta \in (0,1)$. By Lemma 6.9, there exists a constant $c > 0$ such $1 - e^{-x} \geq x - cx^\delta$ for all $x \geq 0$. Thus,

$$\begin{aligned}1 - u_\lambda(t) &= \mathbb{E}[1 - e^{-\lambda \mathcal{X}(t)}] \geq \mathbb{E}(\lambda \mathcal{X}(t) - \lambda^\delta \mathcal{X}(t)^\delta) \\ &= \lambda \mathbb{E}(\mathcal{X}(t)) - \lambda^\delta \mathbb{E}(\mathcal{X}(t)^\delta) \\ &= \lambda \eta(t) - \lambda^\delta \mathbb{E}(\mathcal{X}(t)^\delta). \end{aligned} \tag{6.23}$$

By (3.6) and Proposition 6.7, part *(iii)*, we have

$$\mathbb{E}(\mathcal{X}_n(t)^\delta) \le \frac{\eta_\delta(t)}{2^\delta} \le Ct^{-\gamma\delta} \quad \forall t > 0,\, n \in \mathbb{N}.$$

Substituting this estimate into (6.23), we get

$$\liminf_{n\to\infty} \frac{1-u_\lambda(t)}{t^{-\gamma}} \ge \liminf_{n\to\infty} \left(\lambda\frac{\eta(t)}{t^{-\gamma}} - C\frac{t^{-\gamma\delta}}{t^{-\gamma}}\right) = \lambda \lim_{n\to\infty} \frac{\eta(t)}{t^{-\gamma}} = \lambda.$$

This completes the proof. □

Lemma 6.9. *For each* $\delta \in (1,2)$*, there exists* $c_\delta > 0$ *such that*

$$1 - e^{-x} \ge x - c_\delta x^\delta \quad \forall\, x > 0$$

Proof. For a fixed $c > 0$, let $f(x) = 10e^{-x} - (x - cx^\delta)$. Then

$$\begin{aligned} f'(x) &= e^{-x} - 1 + c\delta x^{\delta-1} \\ f''(x) &= -e^{-x} + c\delta(\delta-1)x^{\delta-2} \end{aligned}$$

Since $\delta \in (1,2)$, one can choose c sufficiently large such that $f''(x) > 0$ for all $x > 0$. Then $f'(x) \ge f'(0) = 0$ for all $x > 0$. Then $f(x) \ge f(0) = 0$ for all $x > 0$. □

6.3 Proof of Theorem 6.1 for $u_0 \in R_\alpha$

According to Proposition 5.4, parts *1(ii)* and *2(ii)* (see also [13, Prop. 2.2], [14, Prop. 4.1]), when $u_0 \ge 0$, the stochastic Picard iterations scheme for the α-Riccati model with the ground state $\overline{X}_0(t) \equiv 1$, i.e.,

$$\overline{X}_0(t) \equiv 1, \quad \overline{X}_n(t) = \begin{cases} u_0 & \text{if} \quad T_\theta \ge t, \\ \overline{X}^{(1)}_{n-1}(\alpha(t-T_\theta))\overline{X}^{(2)}_{n-1}(\alpha(t-T_\theta)) & \text{if} \quad T_\theta < t, \end{cases}$$

where $\overline{X}^{(1)}_{n-1}$ and $\overline{X}^{(2)}_{n-1}$ are, conditionally on T_θ, i.i.d. copies of $\overline{X}_{n-1}$, almost surely has a limit – a maximal solution process $\overline{X}(t)$, satisfying (2.16). Moreover, if $u_0 \in R_\alpha$, then $\overline{u}(t) = \mathbb{E}(\overline{X}(t)) < \infty$ is a solution to (1.2), where R_α is defined by (6.2).

Let $\lambda > 0$. Consider $X_{M,\delta,\lambda}(t)$ be the process defined in Theorem 6.5 (defined for $\alpha \in (0,1]$), and the process $X_\lambda(t)$ is the process defined in Theorem 6.8 (defined for $\alpha > 2$). For each $\lambda \ge 0$, and $\alpha > 1$ consider the following solution process (satisfying (2.16) with $u_0 = 1$):

$$X^\lambda_{u_0=1}(t) = \begin{cases} X_{M,\delta,\lambda}(t) & \text{if} \quad 1 < \alpha \le 2, \\ X_\lambda(t) & \text{if} \quad \alpha > 2. \end{cases}$$

By Theorem 6.5 and Theorem 6.8, for each $\lambda > 0$, the function $u^*_\lambda(t) = \mathbb{E}(X^\lambda_{u_0=1}(t))$ satisfies

$$\frac{1-u^*_\lambda(t)}{t^{-\gamma}} = \lambda.$$

Let $u_0 \in R_\lambda$. Note that process $X_\lambda(t) = \overline{X}(t) X^\lambda_{u_0=1}(t)$ satisfies (2.16), i.e.

$$X_{\lambda,u_0}(t) = \begin{cases} u_0 & \text{if } \ T_\theta \geq t, \\ X^{(1)}_{\lambda,u_0}(\alpha(t - T_\theta)) X^{(2)}_{\lambda,u_0}(\alpha(t - T_\theta)) & \text{if } \ T_\theta < t. \end{cases}$$

(This can be seen by multiplying both sides of (2.16) with $u_0 = 1$ with corresponding sides of (2.16) with $u_0 = 1$.)

Thus, $u_{\lambda,u_0}(t) = \mathbb{E}(X_{\lambda,u_0}(t))$ satisfies (1.2) and $0 \leq u_{\lambda,u_0}(t) \leq \overline{u}(t)$. Note that $\overline{X}(t) = 1$ on the event $[L < t]$. Thus,

$$\begin{aligned} u_{\lambda,u_0}(t) = \mathbb{E}(\overline{X}(t) X^\lambda_{u_0=1}(t)) &= \mathbb{E}(X^\lambda_{u_0=1}(t)) - \mathbb{E}((1 - \overline{X}) X^\lambda_{u_0=1}(t)) \\ &= \mathbb{E}(X^\lambda_{u_0=1}(t)) - \mathbb{E}((1 - \overline{X}) X^\lambda_{u_0=1}(t) \mathbb{1}_{[L>t]}) \end{aligned}$$

and

$$\frac{1 - u_{\lambda,u_0}(t)}{t^{-\gamma}} = \frac{1 - u^*_\lambda(t)}{t^{-\gamma}} + t^\gamma \mathbb{E}((1 - \overline{X}) X^\lambda_{u_0=1}(t) \mathbb{1}_{[L>t]}).$$

In the case $u_0 \in R_\alpha$, $u_0 > 1$, we have $\alpha > 5/2$ and it follows from [14, Remark 3.3 and Thm 4.2] that $\overline{u}(t) = \mathbb{E}(\overline{X}(t)) \leq 1 + ce^{-t}$. Since we also have $\overline{X}(t) \geq 1$ when $u_0 > 1$, we conclude that

$$0 \leq \mathbb{E}((\overline{X} - 1) X^\lambda_{u_0=1}(t) \mathbb{1}_{[L>t]}) \leq \mathbb{E}(\overline{X} - 1) = \overline{u}(t) - 1 \leq ce^{-t}.$$

In the case $u_0 \in [0, 1]$ from definition of $\overline{X}$, in the case, $\overline{u}(t) = \mathbb{E}(\overline{X}(t)) \in [0, 1]$, and so

$$0 \leq \mathbb{E}((1 - \overline{X}) X^\lambda_{u_0=1}(t) \mathbb{1}_{[L>t]}) \leq t^\gamma \mathbb{P}(L > t) \leq Ce^{-t}.$$

Thus, for all $u_0 \in R_\alpha$

$$t^\gamma \mathbb{E}((1 - \overline{X}) X^\lambda_{u_0=1}(t) \mathbb{1}_{[L>t]}) \to 0, \quad \text{as } t \to \infty$$

Therefore,

$$\frac{1 - u_{\lambda,u_0}(t)}{t^{-\gamma}} = \lambda,$$

which finishes the proof of Theorem 6.1.

6.4 Alternative proof of Theorem 6.1 for $u_0 = 0$

In the case $u_0 = 0$, (1.2) has a minimal solution $u \equiv 0$ and a maximal solution $u(t) = \mathbb{E}\overline{X}(t)$. Athreya [2, Thm 2] uses the Picard's iteration

$$u_{(0)}(t) = e^{-t^{-\gamma}}, \quad u_{(n)}(t) = \int_0^t e^{-(t-s)} u^2_{(n-1)}(\alpha s) ds$$

to derive a third solution to (1.2). He shows that the limit function $u(t) = \lim_{n\to\infty} u_{(n)}(t)$ satisfies $\liminf_{t\to\infty} t^\gamma (1 - u(t)) \geq 1$. Below, we will show that one can use the stochastic Picard iterations with the ground state

$$\mu(t) = e^{-t^{-\gamma}} \mathbb{1}_{t>0}$$

to generate infinitely many solutions to (1.2).

Proposition 6.10. *The sequence of stochastic processes*

$$X_n(t) = \prod_{|v|=n-1} \mu^2(\alpha^n(t-\Theta_v)), \quad \forall\, n \geq 1.$$

converges almost surely to a solution process $X(t)$ *as* $n \to \infty$*. Moreover,* $u_\lambda(t) = \mathbb{E}(X(t)^\lambda)$ *satisfies* (1.2) *and*

$$\lim_{t\to\infty} \frac{1-u_\lambda(t)}{t^{-\gamma}} = \lambda.$$

Proof. Following the the approach form the proof of Proposition 6.3 One can rewrite $X_n(t)$ as

$$\begin{aligned} X_n(t) &= \prod_{|v|=n-1} \exp\left(-2(\alpha^n(t-\Theta_v))^{-\gamma}\right) \mathbb{1}_{[L_n<t]} \\ &= \prod_{|v|=n-1} \exp\left(-\frac{(t-\Theta_v)^{-\gamma}}{2^{n-1}}\right) \mathbb{1}_{[L_n<t]} = e^{-M_n(t)} \mathbb{1}_{[L_n<t]}, \end{aligned}$$

where

$$M_n(t) = \frac{1}{2^{n-1}} \sum_{|v|=n-1} (t-\Theta_v)^{-\gamma}, \quad L_n = \sup_{|v|=n} \sum_{j=0}^{n} \frac{T_{v|j}}{\alpha^j}.$$

On the event $[L>t]$, $X_n(t) = 0$ for sufficiently large n. On the event $[L<t]$,

$$\begin{aligned} M_{n+1}(t) &= \frac{1}{2^n} \sum_{|w|=n} (t-\Theta_w)^{-\gamma} \\ &= \frac{1}{2^n} \sum_{|v|=n-1} \left((t-\Theta_v-\alpha^{-n}T_{v1})^{-\gamma} + (t-\Theta_v-\alpha^{-n}T_{v2})^{-\gamma}\right) \\ &> \frac{1}{2^n} \sum_{|v|=n-1} \left((t-\Theta_v)^{-\gamma} + (t-\Theta_v)^{-\gamma}\right) \\ &= \frac{1}{2^{n-1}} \sum_{|v|=n-1} (t-\Theta_v)^{-\gamma} = M_n(t). \end{aligned}$$

Thus, the sequence $M_n(t)$ is increasing on the event $[L<t]$. Thus, a limit $M(t) = \lim_{n\to\infty} M_n(t)$ exists almost surely. Then, $X_n(t)$ converges almost surely to $X(t) = e^{-M(t)}\mathbb{1}_{[L<t]}$. Note that

$$\frac{1}{2^{n-1}} \sum_{|v|=n-1} (t-S_n)^{-\gamma} \leq M_n(t) \leq \frac{1}{2^{n-1}} \sum_{|v|=n-1} (t-L_n)^{-\gamma}.$$

In other words,

$$(t-S_n)^{-\gamma} \leq M_n(t) \leq (t-L_n)^{-\gamma}$$

Thus,

$$(t-S)^{-\gamma}\mathbb{1}_{[L<t]} \leq M(t)\mathbb{1}_{[L<t]} \leq (t-L)^{-\gamma}\mathbb{1}_{[L<t]}. \tag{6.24}$$

Consequently, $0 < X(t) < 1$ on the event $[L < t]$. As in the proof of Proposition 6.3, we have that $X_n(t)$ are stochastic Picard iterations of $X_0(t) = \mu(t)$, i.e.

$$X_n(t) = \begin{cases} 0 & \text{if} \quad T_\theta \geq t, \\ X_{n-1}^{(1)}(\alpha(t - T_\theta))X_{n-1}^{(2)}(\alpha(t - T_\theta)) & \text{if} \quad T_\theta < t. \end{cases}$$

where $X_{n-1}^{(1)}$ and $X_{n-1}^{(2)}$ are conditionally on T_θ i.i.d. copies of X_{n-1}. Thus,

$$X(t) = \begin{cases} 0 & \text{if} \quad T_\theta \geq t, \\ X^{(1)}(\alpha(t - T_\theta))X^{(2)}(\alpha(t - T_\theta)) & \text{if} \quad T_\theta < t. \end{cases}$$

where $X^{(1)}$ and $X^{(2)}$ are i.i.d. copies of X. For each $\lambda > 0$, $u_\lambda(t) = \mathbb{E}(X(t)^\lambda)$ solves (1.2). From here, one can use the same estimating technique used in the proof of Proposition 6.4 (ii) to show that

$$\lim_{t\to\infty} \frac{1 - u_\lambda(t)}{t^{-\gamma}} = \lambda.$$

□

7 Numerical Simulations

In this section, we present numerical results based on Monte Carlo simulation to experimentally illustrate the construction of the multiple solutions to the α-Riccati equations, particularly by simulating, for various values of $\lambda > 0$ solution processes $X_{M,\delta,\lambda}(t)$ – defined in Theorem 6.5 (for $\alpha \in (0, 1]$), and the solution processes $X_\lambda(t)$ – defined in Theorem 6.8 (for $\alpha > 2$), via the use of stochastic Picard iterations.

In the case $0 < \alpha \leq 2$, the ground state is given by (6.5), i.e.

$$X_0(t) = \begin{cases} 1 & \text{if} \quad t \leq M, \\ 1 - \delta t^{-\gamma} & \text{if} \quad t > M \end{cases}$$

where $M > 0$ is sufficiently large and $\delta > 0$ is sufficiently small. Recall that by Theorem 6.5 that the corresponding solution has an asymptotic behavior

$$\lim_{t\to\infty} \frac{1 - u_\lambda(t)}{\lambda\delta t^{-\gamma}} \quad = \quad 1. \tag{7.1}$$

For numerical simulation, we choose $\delta = 0.5$ and $M = 10$. We divide the time interval into the lower range $0 \leq t \leq 4$, middle range $4 < t < 64$, and upper range $64 < t < 256$, and subdivide each range with equi-distance grid points in log-space. We use 20, 15, 10 grid points for lower, middle, upper range, respectively. At each grid point t, we generate the process $X_{n=8}$ with a sample size of $N = 200$. For each realization of X_8, we compute X_8^λ with $\lambda = 1, 3, 9$ and approximate $u_\lambda(t) \approx \mathbb{E}X_8^\lambda(t)$. To check the convergence rate (7.1), we plot compare the log-log plot of u_λ with the log-log plot of $\lambda\delta t^{-\gamma}$ (which is a straight line).

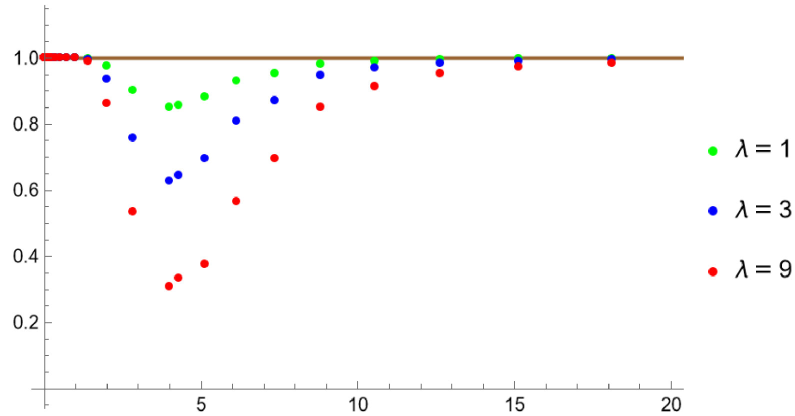

Figure 3: Discrete graph of u_λ for $\lambda = 1, 3, 9$ and $\alpha = 1.4$

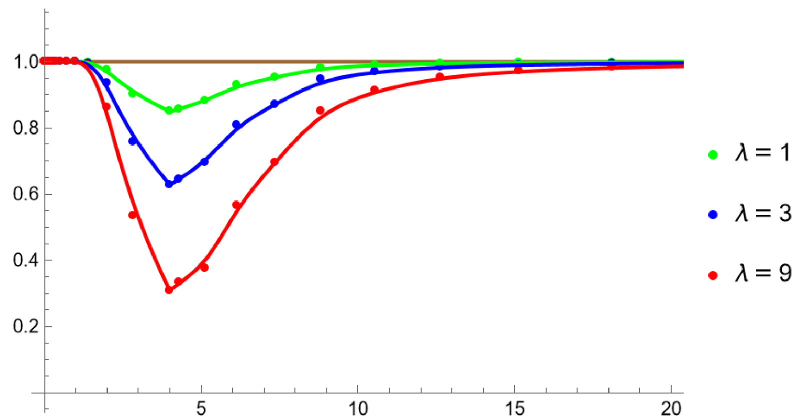

Figure 4: Interpolated graph of u_λ for $\lambda = 1, 3, 9$ and $\alpha = 1.4$

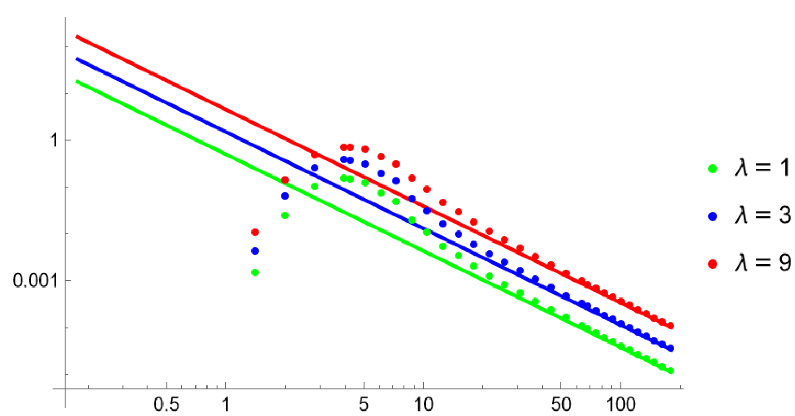

Figure 5: Log-log plot of $\lambda\delta t^{-\gamma}$ and Log-log plot of $1 - u_\lambda$ for $\lambda = 1, 3, 9$ and $\alpha = 1.4$

In the case $\alpha > 2$, the ground state is $X_0(t) = e^{-\eta(t)}$ where $\eta(t) = \mathbb{E}(\tilde{\mathfrak{X}}_*(t))$, see Theorem 3.5. On can show using indiction that $\tilde{\mathfrak{X}}_0$ is a limit of stochastic Picard iterations with the ground state $\tilde{\mathfrak{X}}_0(t) = t^{-\gamma}\mathbb{1}_(t > 0)$:

$$\tilde{\mathfrak{X}}_0(t) = t^{-\gamma}, \quad \tilde{\mathfrak{X}}_n(t) = \begin{cases} 0 & \text{if} \quad T_0 \geq t \\ 2\tilde{\mathfrak{X}}^{(1)}_{n-1}(\alpha(t - T_0)) & \text{if} \quad T_0 < t, \end{cases}$$

and by a uniform integrability argument, $\eta(t) = \lim_{n\to\infty} \mathbb{E}(\tilde{\mathfrak{X}}_n(t))$. Recall that by Theorem 6.8,

$$\lim_{t\to\infty} \frac{1 - u_\lambda(t)}{\lambda t^{-\gamma}} = 1. \tag{7.2}$$

To avoid technical difficulties connected to approximating η before being able to simulate $X_{n=8}$, we use the fact that $\eta_n(t) = \mathbb{E}(\tilde{\mathfrak{X}}_n(t))$ satisfies a deterministic Picard iteration

$$\eta_0(t) = t^{-\gamma}, \quad \eta_n(t) = \int_0^t 2e^{-s}\eta_{n-1}(\alpha(t - s))ds$$

One can use Mathematica to get an explicit formula for η_1, η_2, η_3,... However, these formula exhibit numerical artifacts for large values of t. They collapse to 0 instead of decaying as $t^{-\gamma}$. To

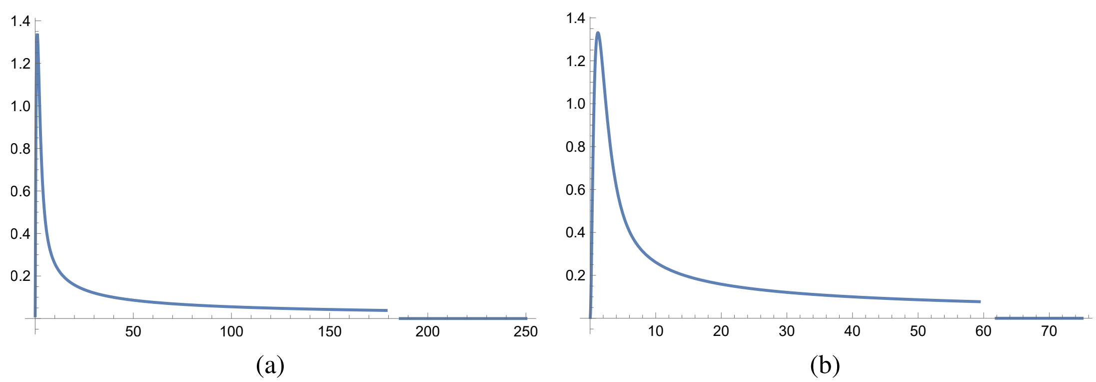

Figure 6: Graphs of η_2 (a) and of η_3 (b) with numerical artifacts

guarantee that η decays as $t^{-\gamma}$, we will use the following approximations:

$$\eta_1(t) \approx \tilde{\eta}_1(t) = \begin{cases} \int_0^t 2e^{-s}\eta_0(\alpha(t - s)) & \text{if} \quad t < 50 \\ t^{-\gamma} & \text{if} \quad t > 50 \end{cases}$$

$$\eta_2(t) \approx \tilde{\eta}_2(t) = \begin{cases} \int_0^t 2e^{-s}\tilde{\eta}_1(\alpha(t - s)) & \text{if} \quad t < 50 \\ t^{-\gamma} & \text{if} \quad t > 50 \end{cases}$$

And then approximate $\eta(t) \approx \tilde{\eta}_2(t)$, which significantly reduces the cost of computation. We will discretize the time interval the same way as in the case $1 < \alpha \leq 2$.

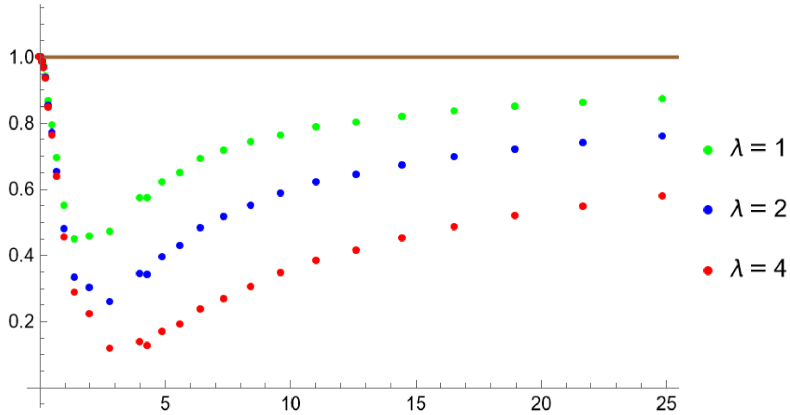

Figure 7: Discrete graph of u_λ for $\lambda = 1, 2, 4$ and $\alpha = 3$

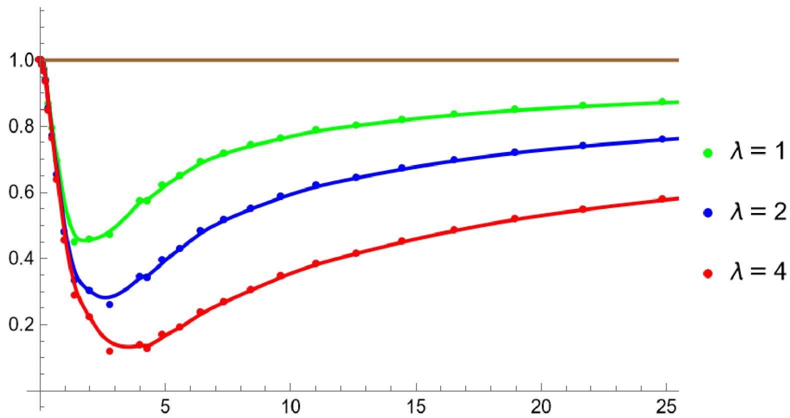

Figure 8: Interpolated graph of u_λ for $\lambda = 1, 2, 4$ and $\alpha = 3$

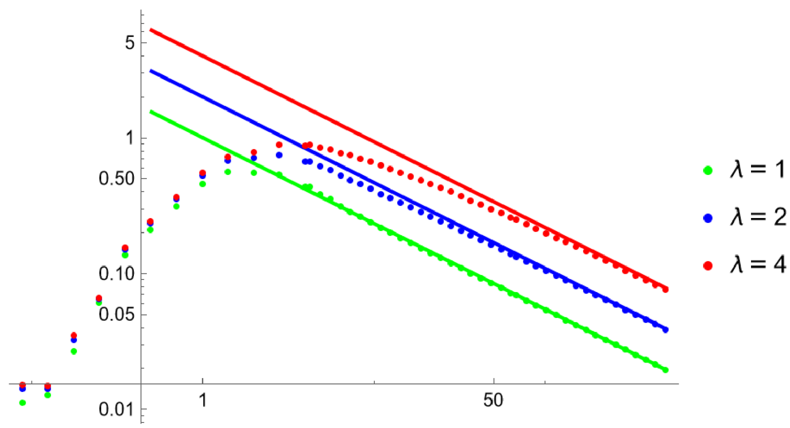

Figure 9: Log-log plot of $\lambda t^{-\gamma}$ and Log-log plot of $1 - u_\lambda$ for $\lambda = 1, 2, 4$ and $\alpha = 3$

References

[1] David Aldous and Paul Shields, *A diffusion limit for a class of randomly-growing binary trees*, Probability Theory and Related Fields **79** (1988), no. 4, 509–542.

[2] K. B. Athreya, *Discounted branching random walks*, Adv. in Appl. Probab. **17** (1985), no. 1, 53–66.

[3] Katharina Best and Peter Pfaffelhuber, *The Aldous-Shields model revisited with application to cellular ageing*, Electronic Communications in Probability **15** (2010), 475–488.

[4] Rabi Bhattacharya and Edward C Waymire, *Continuous parameter Markov processes and stochastic differential equations*, Springer Graduate Texts in Mathematics, 2023.

[5] Leo Breiman, *Probability*, SIAM, 1992.

[6] Philippe Choquard and Joël Wagner, *On the "mean field" interpretation of burgers' equation*, Journal of statistical physics **116** (2004), 843–853.

[7] Radu Dascaliuc, Nicholas Michalowski, Enrique Thomann, and Edward C. Waymire, *Symmetry breaking and uniqueness for the incompressible Navier-Stokes equations*, Chaos **25** (2015), no. 7, 075402, 16.

[8] ______, *A delayed Yule process*, Proc. Amer. Math. Soc. **146** (2018), no. 3, 1335–1346.

[9] Radu Dascaliuc, Nicholas Michalowski, Enrique Thomann, and Edward C Waymire, *Complex Burgers equation: A probabilistic perspective*, Sojourns in probability theory and statistical physics-i, 2019, pp. 138–170.

[10] Radu Dascaliuc, Tuan N. Pham, and Enrique Thomann, *On Le Jan-Sznitman's stochastic approach to the Navier-Stokes equations*, Trans. Amer. Math. Soc. **377** (2024), no. 4, 2335–2365.

[11] Radu Dascaliuc, Tuan N. Pham, Enrique Thomann, and Edward C. Waymire, *Doubly stochastic Yule cascades (Part I): The explosion problem in the time-reversible case*, J. Funct. Anal. **284** (2023), no. 1, Paper No. 109722, 25.

[12] ______, *Doubly stochastic Yule cascades (part II): The explosion problem in the non-reversible case*, Ann. Inst. Henri Poincaré Probab. Stat. **59** (2023), no. 4, 1907–1933.

[13] ______, *Erratum to "Stochastic explosion and non-uniqueness for α-Riccati equation" [J. Math. Anal. Appl. 476 (1) (2019) 53–85]*, J. Math. Anal. Appl. **527** (2023), no. 2, Paper No. 127420, 6.

[14] Radu Dascaliuc, Enrique A. Thomann, and Edward C. Waymire, *Stochastic explosion and non-uniqueness for α-Riccati equation*, J. Math. Anal. Appl. **476** (2019), no. 1, 53–85.

[15] David S. Dean and Satya N. Majumdar, *Phase transition in a generalized Eden growth model on a tree*, J. Stat. Phys. **124** (2006), no. 6, 1351–1376.

[16] David S Dean and Satya N Majumdar, *Phase transition in a generalized eden growth model on a tree*, Journal of statistical physics **124** (2006), no. 6, 1351–1376.

[17] William Feller, *An introduction to probability theory and its applications, volume 2*, Vol. 81, John Wiley & Sons, 1991.

[18] Tosio Kato and John B McLeod, *The functional-differential equation $y'(x) = ay(\lambda x) + by(x)$*, Bulletin of the American Mathematical Society **77** (1971), no. 6, 891–937.

[19] Y. Le Jan and A. S. Sznitman, *Stochastic cascades and 3-dimensional Navier-Stokes equations*, Probab. Theory Related Fields **109** (1997), no. 3, 343–366.

[20] Russell Lyons and Yuval Peres, *Probability on trees and networks*, Cambridge Series in Statistical and Probabilistic Mathematics, vol. 42, Cambridge University Press, New York, 2016.

[21] Charles M Newman, *Percolation theory: A selective survey of rigorous results*, Advances in multiphase flow and related problems (1986), 147–167.

[22] Gennady Samorodnitsky, *Extreme value theory, ergodic theory and the boundary between short memory and long memory for stationary stable processes*, The Annals of Probability **32** (2004), no. 2, 1438–1468.

[23] Asaf Shapira and Mykhaylo Tyomkyn, *Quasirandom graphs and the pantograph equation*, The American Mathematical Monthly **128** (2021), no. 7, 630–639.